Math 1b Section 2 Midterm #1, 2/14/06, Solutions

1. Substitute $u = \sqrt{e^x + 1}$. Then $u^2 = e^x + 1$, so $2u\,du = e^x\,dx = (u^2 - 1)\,dx$, so $dx = 2u\,du/(u^2 - 1)$. Thus

$$\int_{\ln 3}^{\ln 8} \frac{dx}{\sqrt{e^x + 1}} = \int_2^3 \frac{2\,du}{u^2 - 1}.$$ 

To evaluate the integral on the right we use partial fractions:

$$\int_2^3 \frac{2\,du}{u^2 - 1} = \int_2^3 \frac{1}{u - 1} - \frac{1}{u + 1}\,du = \ln \left( \frac{u - 1}{u + 1} \right) \bigg|_2^3 = \ln(3/2).$$

2. The surface area is

$$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.$$ 

We calculate

$$1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2},$$ 

so

$$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} 2\pi \frac{2 + \sqrt{1-x^2}}{\sqrt{1-x^2}} \, dx.$$ 

To evaluate this integral we substitute $x = \sin \theta$ for $-\pi/2 \leq \theta \leq \pi/2$. Then $\sqrt{1-x^2} = \cos \theta$ and $dx = \cos \theta \, d\theta$, so

$$A = \int_{-\pi/4}^{\pi/4} 2\pi(2 + \cos \theta) \, d\theta = 2\pi^2 + 2\sqrt{2}\pi.$$ 

3. (a) By definition, the midpoint approximation for $n = 2$ is

$$M_2 = \frac{6 - \frac{2}{2}}{2} \left( \frac{1}{3} + \frac{1}{5} \right) = \frac{16}{15}.$$ 

(b) If $f(x) = 1/x$ and $x > 0$ then $|f''(x)| = 2/x^3$. This is a decreasing function of $x$ (since its derivative is negative), so if $x \geq 2$ then $|f''(x)| \leq 2/2^3 = 1/4 = K$. We use the theorem asserting that the error in the midpoint approximation is bounded by

$$|E_M| \leq \frac{K(b - a)^3}{24n^2} = \frac{(1/4)(6 - 2)^3}{24n^2} = \frac{2}{3n^2}. $$
We want to choose $n$ large enough so that
\[ \frac{2}{3n^2} < \frac{1}{100}. \]
We can take $n = 9$ because $2/243 < 1/100$. (Any $n \geq 9$ is guaranteed to work by the theorem. Some smaller values of $n$ might also work but this is not guaranteed by the theorem.)

4. The integral is convergent. To compute it we use partial fractions:

\[
\int_{2}^{\infty} \frac{dx}{x(x-1)} = \lim_{t \to \infty} \int_{2}^{t} \left( \frac{1}{x-1} - \frac{1}{x} \right) dx = \lim_{t \to \infty} \ln \left( \frac{x-1}{x} \right) \bigg|_{2}^{t}
\]
\[
= \lim_{t \to \infty} \left( \ln \left( \frac{1}{t} \right) - \ln(1/2) \right) = \ln(2).
\]

5. We use integration by parts. We take $u = \arctan x$ and $dv = x^2 \, dx$. Then $du = dx/(1 + x^2)$ and $v = x^3/3$, so

\[
\int x^2 \arctan x \, dx = \frac{x^3 \arctan x}{3} - \frac{1}{3} \int \frac{x^3}{1 + x^2} \, dx.
\]

To evaluate the integral on the right, we divide polynomials to obtain

\[
\int \frac{x^3}{1 + x^2} \, dx = \int \left( x - \frac{x}{1 + x^2} \right) \, dx = \frac{x^2}{2} - \frac{1}{2} \ln(1 + x^2) + C.
\]

Putting this all together, we get

\[
\int x^2 \arctan x \, dx = \frac{x^3 \arctan x}{3} - \frac{x^2}{6} + \frac{1}{6} \ln(1 + x^2) + C.
\]

6. The integral is divergent, by the comparison test. If $0 < x \leq 1$ then

\[
\frac{x + e^x}{x^{3/2}} \geq \frac{e^x}{x^{3/2}} \geq \frac{1}{x^{3/2}} > 0.
\]

The second inequality holds because $e^x$ is an increasing function of $x$ (because its derivative is positive), so $e^x \geq e^0$ when $x \geq 0$. We know that

\[
\int_{0}^{1} \frac{1}{x^{3/2}} \, dx
\]

diverges because $3/2 \geq 1$. So by the comparison test, the integral in question diverges.