MATH 142 MIDTERM 2 SOLUTION

1. (10 points) Determine whether the following statements are true or false, no justification is required.

   (1) A path-connected component of a topological space may not be a closed subset.

      True

   (2) The identification space of a Hausdorff space is still Hausdorff.

      False

   (3) Let $X$ be a topological space and $p, q$ be two points in $X$, then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.

      False

   (4) Let $X$, $Y$ be two path-connected topological spaces with isomorphic fundamental groups, then $X$ and $Y$ are homeomorphic to each other.

      False

   (5) Any contractible topological space is connected.

      True
2. (30 points) Let $f : X \to Y$ be an identification map. Suppose that $Y$ is connected, and for each $y \in Y$, $f^{-1}(y) \subseteq X$ is a connected subspace of $X$. Show that $X$ is a connected space.

Proof. Suppose that $X = U \cup V$ with $U \cap V = \emptyset$, both $U$ and $V$ are open.

For each $y \in Y$, $f^{-1}(y)$ is not empty ($f$ is an onto map). We have $f^{-1}(y) = (f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap V)$ with both $f^{-1}(y) \cap U$ and $f^{-1}(y) \cap V$ are open sets in $f^{-1}(y)$ (under subspace topology), and $(f^{-1}(y) \cap U) \cap (f^{-1}(y) \cap V) = \emptyset$. Since $f^{-1}(y)$ is connected, we have that either $f^{-1}(y) \subseteq U$ or $f^{-1}(y) \subseteq V$ holds and only one of them happens.

Define two subsets $U'$ and $V'$ of $Y$ by $U' = \{ y \in Y \mid f^{-1}(y) \subseteq U \}$ and $V' = \{ y \in Y \mid f^{-1}(y) \subseteq V \}$. Since for any $y \in Y$, either $f^{-1}(y) \subseteq U$ or $f^{-1}(y) \subseteq V$ and only one of them happens, we have $U' \cup V' = Y$ and $U' \cap V' = \emptyset$. Moreover, since $U = f^{-1}(U')$ and $V = f^{-1}(V')$, $U, V$ are open subsets of $X$ and $f : X \to Y$ is an identification map, $U'$ and $V'$ are open sets in $Y$.

Since $Y$ is connected, we have that either $U'$ or $V'$ is empty. Since $f$ is onto, $U = f^{-1}(U')$ and $V = f^{-1}(V')$, we have that either $U$ or $V$ is empty. So $X$ is connected. □
3. (30 points) Let $G$ be a path-connected topological group and $X$ be a path-connected topological space, with $G$ acts on $X$ (as a group of homeomorphisms). For each $x \in X$, we can define a continuous function $i_x : G \to X$, with $i_x(g) = g(x)$ for any $g \in G$.

Show that the kernel of $(i_x)_* : \pi_1(G, e) \to \pi_1(X, x)$ is independent of $x \in X$ (i.e. ker $(i_x)_* = \text{ker } (i_y)_*$ for any $x, y \in X$).

Proof. For $\langle \alpha \rangle \in \pi_1(G, e)$, if it lies in the kernel of $(i_x)_* : \pi_1(G, e) \to \pi_1(X, x)$, then the path $i_x \circ \alpha : I \to X$ defined by $i_x \circ \alpha(s) = \alpha(s)(x)$ satisfies $(i_x \circ \alpha) = e \in \pi_1(X, x)$.

Since $X$ is path connected, there exists a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma_* : \pi_1(X, x) \to \pi_1(X, y)$ defined by $\gamma_*([\beta]) = \langle \gamma^{-1} \cdot \beta \cdot \gamma \rangle$ for any $[\beta] \in \pi_1(X, x)$ is an isomorphism. Since $\langle i_x \circ \alpha \rangle = e \in \pi_1(X, x)$, we have that $e = \gamma_*\alpha = \gamma_*([i_x \circ \alpha]) = \langle \gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma \rangle \in \pi_1(X, y)$.

To show that $\langle \alpha \rangle$ lies in the kernel of $(i_y)_* : \pi_1(G, e) \to \pi_1(X, y)$, we need only to show that $i_y \circ \alpha$ is homotopic to $\gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma$ relative to $\{0, 1\}$. The homotopy $F : I \times I \to X$ from $i_y \circ \alpha$ to $\gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma$ is defined by

$$F(s, t) = \begin{cases} 
\gamma(1 - 3s) & s \in [0, \frac{4}{7}] \\
\alpha(\frac{3s - 1}{2 - 2t})(\gamma(1 - t)) & s \in [\frac{4}{7}, 1 - \frac{4}{7}]
\gamma(3s - 2) & s \in [1 - \frac{4}{7}, 1].
\end{cases}$$

Then $F_0(s) = \alpha(s)(\gamma(1)) = \alpha(s)(y) = i_y \circ \alpha(s)$, and

$$F_1(s) = \begin{cases} 
\gamma(1 - 3s) & s \in [0, \frac{1}{7}]
\alpha(3s - 1)(x) & s \in [\frac{1}{7}, \frac{2}{7}] = (\gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma)(s).
\gamma(3s - 2) & s \in [\frac{2}{7}, 1].
\end{cases}$$

So we have that for any element lies in the kernel of $(i_x)_*$, it lies in the kernel of $(i_y)_*$. By switching $x$ and $y$, we get that any element lies in the kernel of $(i_y)_*$ also lies in the kernel of $(i_x)_*$. So the kernel does not depend on the base point $x$. □
4. (30 points) For a continuous function \( f : S^1 \to S^1 \), show that either there exists \( e^{i\theta} \in S^1 \) such that \( f(e^{i\theta}) = -e^{i\theta} \), or there exists \( e^{i\phi} \in S^1 \) such that \( f(e^{i\phi}) = -e^{-i\phi} \).

(Hint: If \( f(e^{i\theta}) \neq -e^{i\theta} \) and \( f(e^{i\phi}) \neq -e^{-i\phi} \) for any \( e^{i\theta} \in S^1 \), show that \( f \) is homotopic to both \( e^{i\theta} \to e^{i\theta} \) and \( e^{i\phi} \to e^{-i\phi} \), then try to get a contradiction.)

**Proof.** Suppose that \( f(e^{i\theta}) \neq -e^{i\theta} \) and \( f(e^{i\phi}) \neq -e^{-i\phi} \) for any \( e^{i\theta} \in S^1 \), we can construct two homotopies \( F_1, F_2 : S^1 \times I \to S^1 \) by

\[
F_1(e^{i\theta}, t) = \frac{(1 - t)f(e^{i\theta}) + te^{i\theta}}{||(1 - t)f(e^{i\theta}) + te^{i\theta}||}
\]

and

\[
F_2(e^{i\phi}, t) = \frac{(1 - t)f(e^{i\phi}) + te^{-i\phi}}{||(1 - t)f(e^{i\phi}) + te^{-i\phi}||}.
\]

Then \( F_1 \) gives a homotopy from \( f \) to \( e^{i\theta} \to e^{i\theta} \) and \( F_2 \) gives a homotopy from \( f \) to \( e^{i\phi} \to e^{-i\phi} \). Since homotopy is an equivalence relation, we have that \( f_1 : S^1 \to S^1 \) defined by \( f_1(e^{i\theta}) = e^{i\theta} \) and \( f_2 : S^1 \to S^1 \) defined by \( f_2(e^{i\phi}) = e^{-i\phi} \) are homotopic to each other by a homotopy \( G : S^1 \times I \to S^1 \).

Then \( (f_2)_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1) \) is conjugate to \( (f_1)_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1) \) by the path \( \alpha : I \to S^1 \) defined by \( \alpha(t) = G(1, t) \), i.e. \( (f_2)_* = \alpha_* \circ (f_1)_* \). Note that the path \( \gamma : I \to S^1 \) defined by \( \gamma(t) = e^{i2\pi t} \) generates \( \pi_1(S^1, 1) \cong \mathbb{Z} \). Since \( f_1 \circ \gamma = \gamma \) and \( f_2 \circ \gamma = \gamma^{-1} \), the corresponding induced maps on fundamental groups satisfy that \( (f_1)_*(\langle \gamma \rangle) = \langle \gamma \rangle \) and \( (f_2)_*(\langle \gamma \rangle) = \langle \gamma^{-1} \rangle \).

Since \( \pi_1(S^1, 1) \cong \mathbb{Z} \) is abelian, we have \( \langle \alpha^{-1} \rangle \cdot \langle \gamma \rangle \cdot \langle \alpha \rangle = \langle \gamma \rangle \). So \( \langle \gamma^{-1} \rangle = (f_2)_*(\langle \gamma \rangle) = \alpha_*((f_1)_*(\langle \gamma \rangle)) = \alpha_* (\langle \gamma \rangle) = \langle \alpha^{-1} \rangle \cdot \langle \gamma \rangle \cdot \langle \alpha \rangle = \langle \gamma \rangle \). It implies that \( 1 = -1 \) in the group \( \mathbb{Z} \), which is a contradiction.

So either there exists \( e^{i\theta} \in S^1 \) such that \( f(e^{i\theta}) = -e^{i\theta} \), or there exists \( e^{i\phi} \in S^1 \) such that \( f(e^{i\phi}) = -e^{-i\phi} \).