Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

The problems have equal weight. We write $|G|$ for the order of a group $G$.

1. Find the number of conjugates of $(1\ 2\ 3)(4\ 5\ 6)$ in $A_6$. (For this problem, and the ones below, be sure to explain your work in complete English sentences.)

2. Let $p$ be an odd prime, and let $G$ be a dihedral group $D_{2n}$. Show that all $p$-Sylow subgroups of $G$ are cyclic. Find the number of such subgroups.

3. Suppose that $G$ is a finite group and that $H$ is a subgroup of $G$. Let $N = N_G(H)$ be the normalizer of $H$.
   
   a. Let $H_1 = H, H_2, H_3, \ldots, H_k$ be the distinct conjugates of $H$ in $G$. Prove the formula
   
   $$\sum_{i=1}^{k} |H_i| = |H| \cdot (G : N) = |G|/(N : H).$$

   b. If $H \neq G$, show that $\bigcup_{i=1}^{k} H_i \neq G$.

4. Let $G$ be a group (possibly infinite) and let $H$ be a subgroup of $G$ for which the set $G/H$ is finite. Use the action of $G$ by left multiplication on $G/H$ to show that there is a normal subgroup $N$ of $G$ such that $N \subseteq H$ and such that $G/N$ is a finite group.

5. Let $G$ be a group.
   
   a. For each $g \in G$, let $\sigma_g$ be the inner automorphism “conjugation by $g$.” Suppose that $\varphi$ is an automorphism of $G$. Establish the formula $\varphi \sigma_g \varphi^{-1} = \sigma_{\varphi(g)}$.

   b. If $G$ has trivial center and $\varphi$ commutes with all $\sigma_g$, show that $\varphi$ is the identity map.