1. Find the number of conjugates of $(123)(456)$ in $A_6$. (For this problem, and the ones below, be sure to explain your work in complete English sentences.)

In $S_6$, the conjugates of $\sigma = (123)(456)$ are precisely the permutations that have the same cycle type as $\sigma$. To make such a permutation, you lay down the six numbers $1 \to 6$ in some order. There are $6!$ ways to do this, but: each 3-cycle can be written in three ways, and the order of the two 3-cycles doesn’t matter. Accordingly, there are $6!/(3 \cdot 3 \cdot 2) = 40$ such permutations. Still in $S_6$, the centralizer of $\sigma$ thus has 18 elements. Among them is the permutation that swaps 1 with 4, 2 with 5 and 3 with 6. This permutation is $(14)(25)(36)$; it’s odd. Thus the situation is like many that we discussed in class. Namely, when we pass to $A_6$, the group order gets halved but so does the centralizer. Accordingly, $\sigma$ has the same number of conjugates in $A_6$ as it does in $S_6$. Thus the answer is “40.”

2. Let $p$ be an odd prime, and let $G$ be a dihedral group $D_{2n}$. Show that all $p$-Sylow subgroups of $G$ are cyclic. Find the number of such subgroups.

There is a unique $p$-Sylow (which is therefore normal): it’s the $p$-part of the cyclic group generated by $r$, which has order $n$. Specifically, write $n$ as $p^i t$, where $t$ is prime to $p$. Then the $p$-Sylow is the cyclic group generated by $r^t$, which has order $p^t$. [Note: if $i = 0$, one shouldn’t technically speak of the $p$-Sylow subgroup of $G$ because $p$-Sylows are supposed to be non-trivial. If you say that the number of $p$-Sylows is 0 in the case where $p$ doesn’t divide $n$, you’ll get full credit and some extra respect.]

3. Suppose that $G$ is a finite group and that $H$ is a subgroup of $G$. Let $N = N_G(H)$ be the normalizer of $H$.

   a. Let $H_1 = H, H_2, H_3, \ldots, H_k$ be the distinct conjugates of $H$ in $G$. Prove the formula

   \[ \sum_{i=1}^{k} |H_i| = |H| \cdot (G : N) = |G|/(N : H). \]

   All the conjugates have the same number of elements, so the sum is $k \cdot |H|$. How do we know that $k = (G : N)$? It’s a special case of the general rule that the orbit of $x \in X$
is \( G/G_x \) when \( G \) acts on a set \( X \) and \( G_x \) is the stabilizer of an element \( x \) of \( X \). Here, \( X \) is the set of conjugates of \( H \), and the orbit of \( H \) consists of the entire set (by definition). Now \((G:H) = (G:N)(N:H)\) (e.g., because all three indices can be written as fractions in a way that makes this formula obvious). Writing \((G:H) = |G|/|H|\), we get the equality of the middle expression and the expression on the right.

The takeaway here is that the sum on the left is \( \leq |G| \) because the denominator \((N:H)\) is a positive integer.

b. If \( H \neq G \), show that \( \bigcup_{i=1}^{k} H_i \neq G \).

The sum of the sizes of the sets on the left is at most the size of \( G \). Hence the union on the left can be all of \( G \) only if the union is disjoint. But the union isn’t disjoint because 1 (the identity of \( G \)) is in all the groups \( H_i \) and because there are at least two groups \( H_i \) (in view of the assumption that \( H \) isn’t all of \( G \)).

4. Let \( G \) be a group (possibly infinite) and let \( H \) be a subgroup of \( G \) for which the set \( G/H \) is finite. Use the action of \( G \) by left multiplication on \( G/H \) to show that there is a normal subgroup \( N \) of \( G \) such that \( N \subseteq H \) and such that \( G/N \) is a finite group.

The indicated action gives you a homomorphism

\[ \varphi : G \rightarrow S_{G/H}. \]

Let \( N \) be the kernel of \( \varphi \). We have \( N \subseteq H \) because \( N \) is the group of elements of \( G \) that fix all elements of \( G/H \), while \( H \) is the group of elements that fix the coset \( H = 1 \cdot H \) in the set \( G/H \). By the first isomorphism theorem, we have an injection \( G/N \hookrightarrow S_{G/H} \). Since \( G/H \) is a finite set, the symmetric group \( S_{G/H} \) is finite. Thus \( G/N \) is a finite group.

5. Let \( G \) be a group.

a. For each \( g \in G \), let \( \sigma_g \) be the inner automorphism “conjugation by \( g \).” Suppose that \( \varphi \) is an automorphism of \( G \). Establish the formula \( \varphi \sigma_g \varphi^{-1} = \sigma_{\varphi(g)} \).

Let \( x \) be an element of \( G \). We have

\[ (\varphi \sigma_g \varphi^{-1})(x) = \varphi(g \varphi^{-1}(x)g^{-1}) = \varphi(g)\varphi(\varphi^{-1}(x))\varphi(g^{-1}) = \varphi(g)x\varphi(g)^{-1} = \sigma_{\varphi(g)}(x). \]

b. If \( G \) has trivial center and \( \varphi \) commutes with all \( \sigma_g \), show that \( \varphi \) is the identity map.

By part (a), if \( \varphi \) commutes with all \( \sigma_g \), then \( \sigma_g = \sigma_{\varphi(g)} \) for all \( g \in G \). Because \( G \) has trivial center, two elements \( a \) and \( b \) of \( G \) are equal if and only if the automorphisms \( \sigma_a \) and \( \sigma_b \) are equal. Indeed, if \( \sigma_a = \sigma_b \), then you’ll find by messing around that \( ab^{-1} \) commutes with all elements of \( G \) and is therefore the identity. Accordingly, we have \( \varphi(g) = g \) for all \( g \in G \), which shows of course that \( \varphi \) is the identity map.