The problems have equal weight.

1. Suppose that $G$ is a finite group and that $g \in G$ has order $n$ (where $n$ is a positive integer). Let $i$ be an integer. Find a formula for the order of $g^i$ and prove that your formula is correct.

2. Suppose that $H$ is a finite group in which each non-identity element has order 2. Prove that $H$ is abelian.

3. Let $x$ be an element of the dihedral group $D_{2n}$ ($n \geq 3$). Describe explicitly the set of conjugates of $x$ (i.e., the set of elements of the form $gxg^{-1}$). Treat separately the cases where $x$ is a power of $r$ and where $x$ is not a power of $r$.

4. Let $\sigma$ be the 20-cycle $(1 \ 2 \ 3 \ 4 \ \cdots \ 17 \ 18 \ 19 \ 20)$. What are the different cycle types that occur as we consider the various powers of $\sigma$? For which integers $i$ is $\sigma^i$ a 20-cycle?

5. Let $p$ be a prime number. Find the number of invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}/p\mathbb{Z}$. For $t \in (\mathbb{Z}/p\mathbb{Z})^*$, show that the number of such matrices with determinant $t$ is equal to the number of such matrices with determinant 1. What is the latter number?