RIBET'S MATH 110 SECOND MIDTERM, PROBLEMS AND ABBREVIATED SOLUTIONS

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. When you answer questions, write your arguments in complete sentences that explain what you are doing: your paper becomes your only representative after the exam is over.

All vector spaces are finite-dimensional over the field of real numbers or the field of complex numbers.

1. Suppose that $T$ is an invertible linear operator on $V$ and that $U$ is a subspace of $V$ that is invariant under $T$. If $v$ is a vector in $V$ such that $Tv \in U$, show that $v$ is an element of $U$.

Quick solution: Let $u = Tv$. Because the restriction of $T$ to $U$ is invertible, there is a unique $v' \in U$ such that $Tv' = u$. Since $Tv' = Tv$ and $T$ is invertible, we have $v' = v$. Hence we have $v \in U$.

2. Suppose that $T$ is a linear operator on $V$ and that $V$ is an inner-product space. Let $T^*$ be the adjoint of $T$. Show that 0 is an eigenvalue of $T$ if and only if 0 is an eigenvalue of $T^*$.

Quick solution: This problem is the special case of problem 28 where we take $\lambda = 0$. In fact, if we can do this special case, then we get the full statement of problem 28 by replacing $T$ by $T - \lambda I$. To say that 0 is an eigenvalue of an operator is to say that the operator is not invertible. Equivalently, this means that its range is smaller than $V$ and also that its null space is non-zero. Thus $T^*$ has 0 as an eigenvalue if and only if its null space is non-zero, and $T$ has 0 as an eigenvalue if and only if its range is smaller than $V$. To see that these statements are equivalent, we can invoke part (a) of Proposition 6.46 on page 120. Specifically, let $U = T^*$. Then $U$ is $\{0\}$ if and only if $U^\perp = V$. These statements follows from the equations $\{0\}^\perp = V$ (everything is perpendicular to 0), $V^\perp = \{0\}$ (only 0 is perpendicular to everything) and $(U^\perp)^\perp = U$ (6.33 on page 112).

3. Let $T$ be a linear operator on $V$. Suppose that there is a non-zero vector $v \in V$ such that $T^3v = Tv$. Show that at least one of the numbers 0, 1, $-1$ is an eigenvalue of $T$.

Quick solution: Because $v$ is in the null space of $T^3 - T$, this operator is not invertible. However, it is the product $T(T - I)(T + I)$; note that the polynomial $x^3 - x$ factors as $x(x - 1)(x + 1)!$. Because the product is non-invertible, at least one factor is non-invertible. To say that $T$ is non-invertible is to say that 0 is an eigenvalue of $T$. To say that $T - I$ is non-invertible is to say that 1 is an eigenvalue of $T$. To say that $T + I$ is non-invertible is to say that $-1$ is an eigenvalue of $T$. ’null said.

4. Let $U$ be a subspace of the inner-product space $V$, and let $P = P_U$ be the orthogonal projection of $V$ onto $U$. [For $v \in V$, write $v = u + y$ with $u \in U$ and $y \in U^\perp$. Then $Pv = u.$] Show that $P = P^*$.

Quick solution: We need to establish the equality $\langle Pv, w \rangle = \langle v, Pw \rangle$ for $v, w \in V$. Let $v$ and $w$ be in $V$, and write $v = u + x$ as in the statement of the problem. Similarly, write $w = u' + x'$. Then we need to prove $\langle u, u' + x' \rangle = \langle u + x, x' \rangle$. However, the term on the left is $\langle u, u' \rangle + \langle u, x' \rangle = \langle u, u \rangle$ because $u$ and $x'$ are perpendicular. Similarly, the term on the right simplifies to $\langle u, u \rangle$. 
