1. (10 points) Determine whether the following statements are true or false, no justification is required.

(1) For a sequence of real numbers \( (a_n) \), if \( \limsup_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

   True

(2) If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} a_n^2 \) converges.

   False

(3) If a continuous function \( f : \mathbb{R} \to \mathbb{R} \) is bounded, then \( f \) is uniformly continuous on \( \mathbb{R} \).

   False

(4) For a power series \( \sum_{n=1}^{\infty} a_n x^n \), if the radius of convergence equals 1, and the power series converges at \( x = 1 \), then it also converges at \( x = -1 \).

   False

(5) Let \( (f_n) \) be a sequence of continuous functions, with the domain of each \( f_n \) being \( \mathbb{R} \). Suppose that \( (f_n) \) uniformly converges to \( f : \mathbb{R} \to \mathbb{R} \), then \( f \) is also continuous.

   True
2. (30 points) Let \( s \) be a real number and \((s_n)\) be a sequence of real numbers. If for any subsequence \((s_{n_k})\) of \((s_n)\), \((s_{n_k})\) has a subsequence \((s_{n_{k_l}})\) satisfying \(\lim_{l\to\infty} s_{n_{k_l}} = s\), show that \(\lim_{n\to\infty} s_n = s\) holds.

\textit{Proof.} Method 1: For the sequence \((s_n)\), there is a subsequence \((s_{n_k})\) such that \(\lim_{k\to\infty} s_{n_k} = \limsup_{n\to\infty} s_n\). Since the limit of any subsequence of \((s_{n_k})\) equals \(\lim_{k\to\infty} s_{n_k}\), which also equals \(s\) by the assumption. So we have \(\limsup_{n\to\infty} s_n = s\).

The same argument shows that \(\liminf_{n\to\infty} s_n = s\). So \(\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s\), which implies \(\lim_{n\to\infty} s_n = s\).

Method 2: Suppose that \(\lim_{n\to\infty} s_n \neq s\), then there exists \(\epsilon > 0\), such that for any \(N \in \mathbb{N}\), there exists \(n > N\) such that \(|s_n - s| \geq \epsilon\).

First take an arbitrary \(n_1 \in \mathbb{N}\) such that \(|s_{n_1} - s| \geq \epsilon\). For \(n_1 \in \mathbb{N}\), there exists \(n_2 > n_1\) such that \(|s_{n_2} - s| \geq \epsilon\). Then for \(n_2 \in \mathbb{N}\), there exists \(n_3 > n_2\) such that \(|s_{n_3} - s| \geq \epsilon\). By doing this process inductively, we get a subsequence \((s_{n_k})\) of \((s_n)\) such that \(|s_{n_k} - s| \geq \epsilon\) for any \(k \in \mathbb{N}\).

So for any subsequence \((s_{n_{k_1}})\) of \((s_{n_k})\), we have \(|s_{n_{k_1}} - s| \geq \epsilon\) for any \(l \in \mathbb{N}\). This implies that \(\lim_{l\to\infty} s_{n_{k_1}} \neq s\), and we get a contradiction here.

\[\square\]
3. (30 points) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function defined by the following rule. If \( x \) is an irrational number, then we define \( f(x) = 0 \). If \( x \) is a rational number, then \( x \) is uniquely expressed as \( x = \frac{p}{q} \) with \( q \in \mathbb{N} \) and \( p, q \) are relative prime integers (they do not have common factors). In this case we define \( f(x) = \frac{1}{q} \) (for example \( f(0) = f\left(\frac{0}{1}\right) = 1, f(1) = f\left(\frac{1}{1}\right) = \frac{1}{1} = f\left(-\frac{1}{1}\right) = f(-1) \) and \( f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = \frac{1}{2} \)).

Show that \( f \) is continuous at all irrational numbers and discontinuous at all rational numbers.

**Proof.** For any rational point \( x_0 = \frac{p}{q} \), we have \( f(x_0) = f\left(\frac{p}{q}\right) = \frac{1}{q} \). We can choose a sequence of irrational numbers \((x_n)\) with \( \lim_{n \to \infty} x_n = x_0 \) (by taking an arbitrary irrational number \( x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \)). Then since all the \( x_n \) are irrational, \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 0 = 0 \neq \frac{1}{q} = f(x_0) \). So \( f \) is not continuous at \( x_0 \), and \( f \) is discontinuous at all rational numbers.

For any irrational number \( x_0 \), we will show that for any \( \epsilon > 0 \), there exists \( \delta > 0 \), such that for any \( x \in \mathbb{R} \) with \( |x - x_0| < \delta \), we have \( |f(x) - f(x_0)| < \epsilon \). Take a natural number \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \epsilon \), and consider all the rational numbers with denominator smaller than \( n \).

For any natural number \( m < n \), there are at most \( 2m \) rational numbers in \((x_0 - 1, x_0 + 1)\) with denominator equals \( m \). Since for \( 2m + 1 \) different rational numbers with denominator equals \( m \), the maximum minus the minimum is greater or equal to 2. So there are finitely many rational numbers (at most \( n(n-1) \) such numbers) in \((x_0 - 1, x_0 + 1)\) with denominator smaller than \( n \). Since \( x_0 \) is an irrational number, there exists \( \delta \in (0, 1) \) such that there is not a rational number with denominator smaller than \( n \) lying in \((x_0 - \delta, x_0 + \delta)\).

Then for any \( |x - x_0| < \delta \), if \( x \) is irrational, then \( |f(x) - f(x_0)| = |0 - 0| = 0 < \delta \).

If \( x \) is irrational, then \( x = \frac{p}{q} \) with \( p \in \mathbb{N} \) and \( p \geq n \). So \( |f(x) - f(x_0)| = |\frac{1}{q} - 0| = \frac{1}{q} \leq \frac{1}{n} < \delta \). So \( f \) is continuous at all irrational points.

\( \square \)
4. (30 points) Suppose that \( f : [2, \infty) \to \mathbb{R} \) is a uniformly continuous function, show that \( g : [2, \infty) \to \mathbb{R} \) defined by \( g(x) = \frac{f(x)}{x} \) is a bounded function.

Then show that \( g : [2, \infty) \to \mathbb{R} \) is a uniformly continuous function.

(Note that \( f \) may not be differentiable on \((2, \infty)\).

Bonus (5 points): Does \( \lim_{x \to +\infty} g(x) \) always exist? Prove your claim or give a counterexample.

Proof. Since \( f \) is uniformly continuous on \([2, \infty)\), for \( \epsilon = 1 \), there exists \( \delta > 0 \), such that for any \( x, y \in [2, \infty) \) with \( |x - y| < \delta \), we have \( |f(x) - f(y)| < 1 \).

Take \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \delta \). So if \( |x - 2| \leq \frac{1}{n} \) and \( x \in [2, \infty) \), then \( |f(x) - f(2)| < 1 \). An induction argument shows that if \( |x - 2| \leq \frac{m}{n} \) and \( x \in [2, \infty) \), then \( |f(x) - f(2)| < m \). So for any \( x \in [2 + \frac{k}{n}, 2 + \frac{k+1}{n}] \) with \( k \geq 1 \), we have

\[
\frac{|f(x)|}{x} \leq \frac{|f(x) - f(2)| + |f(2)|}{x} \leq \frac{k + |f(2)|}{\frac{k}{n}} = n \frac{k + |f(2)|}{k} \leq n(1 + |f(2)|).
\]

So \( g : [2, \infty) \to \mathbb{R} \) is bounded by \( M = n(1 + |f(2)|) \).

For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( x, y \in [2, \infty) \) with \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \epsilon \). Then for any \( x, y \in [2, \infty) \) with \( |x - y| < \min \{ \delta, \frac{\epsilon}{2M} \} \), we have

\[
|g(x) - g(y)| = \left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| = \left| \frac{yf(x) - xf(y)}{xy} \right| \leq \frac{|yf(x) - xf(y)|}{xy} \leq \frac{|yf(x) - xf(x) + xf(x) - xf(y)|}{xy} \\
\leq \frac{|y - x|}{y} \cdot \frac{|f(x)|}{x} + \frac{|f(x) - f(y)|}{y} \leq \frac{\epsilon}{2M} \cdot M + \frac{\epsilon}{2} = \epsilon.
\]

So \( g \) is uniformly continuous on \([2, \infty)\)

\[ \lim_{x \to \infty} g(x) \] may not exist. For example, let \( f : [2, \infty) \to \mathbb{R} \) be defined by \( f(x) = x \cdot \sin (\log x) \).

Then
\[
f'(x) = \sin (\log x) + x \cdot \frac{1}{x} \cdot \cos (\log x) = \sin (\log x) + \cos (\log x),
\]
and \( |f'(x)| \leq 2 \) for any \( x \in [2, \infty) \). Since \( f' \) exists and bounded in \([2, \infty)\), \( f \) is uniformly continuous on \([2, \infty)\).

However \( \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \sin (\log x) \) does not exists. For example, take \( x_n = e^{(n+\frac{1}{2})\pi} \), then \( \lim_{n \to \infty} x_n = \infty \). However, \( g(x_n) = \sin ((n + \frac{1}{2})\pi) = (-1)^n \) so \( \lim_{n \to \infty} g(x_n) \) does not exist. \( \square \)