Higher Segal-Sugawara, exceptional symmetry + fivebranes.

The Virasoro Lie algebra

$$\mathbb{C} \longrightarrow Vir \longrightarrow Vect(\mathbb{C}^\times)$$

is the universal symmetry algebra in 2d (chiral) CFT.

* This talk is about a higher dimensional enhancement of the Virasoro algebra. This algebra is a universal symmetry of 6d $N = (2,0)$ superconformal field theory — at the level of the holomorphic twist.

* The AGT correspondence, in part, associates to a 6d superconformal field theory a 2d CFT. Under this correspondence our enhanced algebra corresponds exactly with the Virasoro.

* We will discuss enhancements of some mathematical consequences of AGT correspondence involving these exceptional algebras.

① Higher dimensional Segal-Sugawara

The chiral boson CFT is described by a field $b(z)$ of spin 1 w/ OPE

$$b(z)\, b(w) \simeq \frac{1}{(z-w)^2}.$$

Family of conformal structures $\quad c = 1 - 12\lambda^2.$

$$T(z) = \frac{1}{2} :b(z)^2: + \lambda\, \partial b(z).$$

Invariantly: The field b is a holomorphic $(1,0)$-form

$$b \in \Omega^{1,0}(\Sigma), \quad \bar{\partial} b = 0.$$

If we resolve the condition of being holomorphic:

$$b \in \Omega^{1,\cdot}(I) \longleftarrow \boxed{\text{1-shifted Poisson space.}}$$

$$\| $$

$$\begin{array}{ccc} \underline{0} & & \underline{1} \\ \Omega^{0,0} & \xrightarrow{\bar{\partial}} & \Omega^{0,1} \end{array}$$

$$\Pi \in \Omega^{1,\cdot}(I)^{\otimes 2} \longleftrightarrow \Pi = \partial : \Omega^{0,\cdot}(I) \longrightarrow \Omega^{1,\cdot}(I).$$

Quantum mechanics	Batalin-Vilkovisky
associative algebra	BV algebra $(A, \Delta_\hbar, \cdot)$
$\downarrow$ $\hbar \to 0$	$\downarrow$ $\hbar \to 0$
Poisson algebra	(-1)-shifted Poisson algebra
$\|$	$\|$
Fun (phase space)	Fun (fields) $\mathbb{P}_0$-alg.

$*_\hbar \rightsquigarrow \{-,-\}$

$\Delta_\hbar \rightsquigarrow \{-,-\}.$

BV quantization is well-suited for factorization algebras.

A factorization algebra on manifold M is

$$\mathcal{F} : \mathrm{Open}(M) \longrightarrow \mathrm{Vect}^{\otimes} \ \left(\text{or } \mathrm{Ch}^{\otimes}, \dots \right)$$

w/ a "multiplication":

$$U \sqcup V \hookrightarrow W \rightsquigarrow \mathcal{F}(U) \otimes \mathcal{F}(V) \xrightarrow{\mu} \mathcal{F}(W).$$

Thm: [CG] The observables of a classical field theory have the structure of a $\mathbb{P}_0$-factorization algebra.

Quantization (which may be obstructed – "anomalies") yields a BV factorization algebra of quantum observables.

Ex: top^l QFTs yield <u>locally constant</u> factorization alg's.

↓ on $M = \mathbb{R}^n$

$\mathbb{E}_n$- algebras.

[Li, Grady-Li-Li]

Ex: Let (X, ω) be a symplectic manifold. There is a 1-dim^l locally constant fact. alg. Obs s.t.

$$\begin{tikzcd} \mathrm{Obs} \arrow[r, phantom, "\simeq"] \arrow[d, "\hbar \to 0"', "\mathrm{BV}" left] & \mathcal{O}_\hbar(X) \arrow[d] \\ \mathrm{Obs}^{cl} \arrow[r, phantom, "\simeq"] & \mathcal{O}(X) \end{tikzcd}$$

$\mathcal{O}_\hbar(X)$ = Moyal def quantization.

P_0- fact. alg $\rightsquigarrow$ Obs^{cl}

* Back $\Omega^{1,*}(\Sigma)$ w/ its BV structure.

* The BV quantization of this shifted Poisson structure on $\Sigma = \mathbb{C}$ recovers the chiral boson vertex algebra.

* Can represent stress tensor as a local Noether current:

$$\mu \in \mathrm{Vect}(\Sigma) \rightsquigarrow \frac{1}{2}\int_\Sigma b \, i_\mu b + \lambda \int_\Sigma J\mu \cdot b .$$

where $J\mu = J(f\partial_z) = \partial_z f$.

"holomorphic Jacobian"

Recall, the Jacobian of Σ is

$$J_\Sigma = H^1(\Sigma;\mathbb{R}) \,/\, H^1(\Sigma;\mathbb{Z})$$

Moduli of cplx structures on top trivial line bundle.

There is a natural isomorphism:

$$T_{\bar\partial} J_\Sigma \cong H^0(\Sigma, \Omega^1).$$

$\rightsquigarrow$ The chiral boson is naturally a family over the Jacobian variety. When Σ is compact, get a line bundle over J_Σ: the partition f^n is naturally a section of this line bundle.

- <u>Intermediate Jacobian</u>: if X is 3-fold (or generally, an odd-dimensional cplx manifold) then the intermediate Jacobian is

2n+1

$$IJ_X = H^3(X, \mathbb{R}) \big/ H^3(X, \mathbb{Z}). \quad \left(H^{2n+1}(X, \mathbb{R}) \big/ H^{2n+1}(X, \mathbb{Z}) \right)$$

$$\Vert$$

$$\|$$

$$\left\{ \begin{array}{l} \text{cplx structures on the} \\ \text{top}^{\ell} \text{ trivial } \boxed{\text{holomorphic 2-gerbe}} \end{array} \right\}.$$

Associated to this moduli space is a higher dim$^{\ell}$ version of a vertex algebra.

$$T_{\bar{\partial}} \mathcal{IJ}_X \simeq H^{\bullet}(X, \Omega^{2,cl}_X)[1]$$

tangent cplx.

sheaf of closed hol 2-forms.

The (higher) chiral boson on X is the theory w/ fields:

$$\Omega^{2,cl}_X[1] \simeq \underset{\underline{-1}}{\Omega^{2,\bullet}_X} \longrightarrow \underset{\underline{0}}{\Omega^{3,\bullet}_X}. \quad \ni \beta$$

There is a BV structure ; shifted Poisson :

$$\pi \in \Omega^{2,cl}_X[1]^{\otimes 2} \hookleftarrow \pi = \partial : \Omega^{\leq 1, \bullet}_X \longrightarrow \Omega^{\geq 2, \bullet}_X .$$

The corresponding action functional is non-local :

$$S(\beta) = \frac{1}{2} \int_X \beta \, \bar{\partial} \, \partial^{-1} \beta .$$

(In $\dim_{\mathbb{C}} = 1$, have $\frac{1}{2} \int b \, \bar{\partial} \, \bar{\partial}^{-1} b$ as well ...)

<u>Thm</u> [Saberi - W.] The holomorphic twist of the 6d $N = (1,0)$ "tensor multiplet" theory is equivalent to this free BV theory.

- Holomorphic symmetry : The sheaf of holomorphic vector fields on X admits a locally free resolution

$$T_X \xhookrightarrow{\simeq} \Omega^{0,\cdot}(X, T^{1,0})$$

sheaf of dg Lie alg's.

Given $\mu \in \Omega^{0,\cdot}(X, T_X^{1,0})$ we can find an explicit current which realizes the natural symmetry of holomorphic vector fields on $\Omega_X^{2,cl}$ by the Lie derivative :

$$T_\mu = \frac{1}{2}\int_X \beta\, i_X \beta + \cdots$$

* If we resolve the condition that β is a closed 2-form, as above, we must introduce the L_∞ terms

$$\int_X \beta^{3,\cdot}\, i_\mu^2\, \beta^{2,\cdot} + \int_X \beta^{3,\cdot}\, i_\mu^3\, \beta^{3,\cdot}$$

② The higher Virasoro algebra

Classically, this $\mu \longmapsto T_\mu$ defines Hamiltonian action (Noether currents) of the sheaf of holomorphic vector fields on the higher chiral boson.

Quantum, there is an anomaly / central extension of v.f.'s which acts.

Thm [W]: The space of anomalies for the symmetry of holomorphic vector fields is

$$H^1_{loc}\left(\mathrm{Vect}^{hol}(X)\right) \simeq \bigoplus_{k=0}^{2n} H^k(X) \otimes H^{2n+1-k}_{GF}(W_n)$$

X = n-dim^{l} cplx mfld

W_n = Lie alg of formal v.f's on $\hat{D}^n$.

* This result is stated in the case X has smoothly trivializable tangent bundle.

When $X = \mathbb{C}^n$, anomalies $\longleftrightarrow$ $H^{2n+1}_{GF}(W_n) \cong H^{2n+2}(BU(n))$

<u>$n=1$</u>: $H^3_{GF}(W_1) \cong H^4(BU(1)) = \langle c_1^2 \rangle$

$\rightsquigarrow$ Virasoro / Weyl anomaly. As local fnl

$$\mu \longmapsto \int J_\mu \partial J_\mu$$

<u>$n=3$</u>: $H^7_{GF}(W_3) = \langle c_1^4, c_1^2 c_2, c_2^2, c_1 c_3 \rangle$

$\rightsquigarrow$ Three independent anomalies

- $\int \mathrm{Tr}(J_\mu)\, \partial\, \mathrm{Tr}(J_\mu)^3$,
- $\int \mathrm{Tr}(J_\mu)\, \partial\, \mathrm{Tr}(J_\mu)\, \mathrm{Tr}(\partial J_\mu\, \partial J_\mu)$
- $\int \mathrm{Tr}\, J_\mu\, \mathrm{Tr}\left((\partial J_\mu)^3\right)$.
- $\int \mathrm{Tr}(J_\mu \partial J_\mu)\, \mathrm{Tr}(\partial J_\mu\, \partial J_\mu)$

Prop: [w] For the higher chiral boson on $\mathbb{C}^3$, the anomaly is

$$\underset{\substack{\| \\ c_{CB}}}{\mathrm{Td}_8} - \underbrace{\left(\mathrm{Td}\cdot \mathrm{ch}(T)\right)\big|_8}_{\text{universal characteristic class from GRR.}} \in H^8(BU(3)) = H^1_{loc}(\mathrm{Vect}^{hol}_{\mathbb{C}^3}).$$

- The Virasoro algebra is an extension of v.f's on the punctured disk $D^{\times}$.

In higher dimensions the punctured disk $D^n - 0$ is no longer affine, but nevertheless:

$$\mathrm{witt}_n = \Omega^{0,\bullet}(D^n - 0, T^{1,0}) \simeq \mathbb{R}\Gamma(D^n - 0, \mathcal{T})$$

is a dg Lie algebra. Anomalies correspond to central extensions via the residue.

$$H^{2n+2}(BU(n)) \longrightarrow \mathrm{Vir}_n \longrightarrow \mathrm{witt}_n$$

Replace $\int_X$ above by $\oint_{S^{2n+1}} = \mathrm{Res}_{z=0}$.

So, using our result above.

<u>Prop</u>: The space of local operators in the holomorphic twist of the $N=(1,0)$ superconformal theory is a representation of $\mathrm{vir}_{3d}(c=c_{CB})$.

[Specifically, its a vacuum representation. Has the structure of a "higher dim^e" vertex algebra...]

$\rightsquigarrow$ Its character recovers the supersymmetric index / partition fn of 6d theory on $S^5 \times S^1$.

But where is this symmetry coming from???

Thm: [Hahner - Raghavendran - Saberi - W] The holomorphic twist of the $\mathcal{N} = (1,0)$ superconformal (Weyl) multiplet is equivalent to $\mathrm{Vect}^{hol}(\mathbb{C}^3)$.

(A global statement would be nice.)

3 M5 branes and exceptional symmetry

* Theory on M5 brane is $\mathcal{N} = (2,0)$ superconformal, we wish to consider the holomorphic twist of this.
* Twisted 11d supergravity can be placed on 11-manifold of the form

$$(S^1 \text{ or } \mathbb{R}) \times Z$$

$\uparrow$ CY 5

When $Z = \mathbb{C}^5$, we've characterized the twist ([RSW]).

* For studying twisted fivebranes, a natural geometry

$$Z = \mathrm{Tot}\left(\begin{array}{c} K_X^{1/2} \otimes \mathbb{C}^2 \\ \downarrow \\ X \end{array} \right) \quad ; \qquad (*)$$

(Twisted)

where X is an arbitrary complex threefold. Fivebranes wrap the zero section.

<u>Thm</u> [Saberi-W] The holomorphic twist of the thy on a single fivebrane in the geometry $(*)$ has fields

$$\begin{array}{lcc} & \underline{-1} & \underline{0} \\ \underline{\text{even}} & \Omega_X^{2,\cdot} \longrightarrow & \Omega_X^{3,\cdot} \\ \underline{\text{odd}} & K_{\mathbb{C}}^{1/2} \otimes \mathbb{C}^2 \ \varphi & \end{array}$$

The "chiral boson" from before

$\mathcal{N}=(1,0)$ "hypermultiplet"

Q : What (super) extension of holomorphic v.f's is a symmetry?

Thm : [Saberi-W] The holomorphic twist of the $N=(2,0)$ thy has (an explicit) symmetry by the (exceptional) Lie superalgebra

$$\mathfrak{e}(3|6) \left\{ \begin{array}{ll} \underline{\text{ev}} : & \mathrm{Vect}^{hol}(X) \oplus \mathfrak{sl}(2) \otimes \mathcal{O}(X) \\ & \quad\ \ \mu \qquad\qquad\qquad\quad A \\ \underline{\text{odd}} : & \Gamma(X, K^{-1/2} \otimes \mathbb{C}^2) . \end{array} \right.$$

The bosonic currents :

$$\int_X \beta \, i_\mu \beta + \int_X \varphi \, L_\mu \varphi + \int_X \varphi \, \bar{A} \cdot \varphi .$$

* [Hahner-Saberi-Raghavendran-W] This is the twist of the (2,0) Weyl multiplet.

* When $X = \mathbb{C}^3$, this symmetry becomes the exceptional Lie superalgebra discovered by Kac et. al.

$$E(3|6): \quad \underline{\text{even}} \quad \omega_3 \oplus \mathfrak{sl}(2)[[z_1, z_2, z_3]]$$

$$\underline{\text{odd}} \quad \mathbb{C}[[z_1, z_2, z_3]]\{dz_i^{-1/2}\} \otimes \mathbb{C}^2 .$$

* As a consequence: the local operators of the 6d $N=(2,0)$ superconformal theory, in the holomorphic twist, form a representation for $E(3|6)$.

$\rightsquigarrow$ Categorifies the 6d superconformal index.

* But this is only the "positive" part of some bigger symmetry, obtained by looking at currents on $\mathbb{C}^3 \setminus 0$.

* At the quantum level, there is an anomaly, Ⓐ.

$$\text{Ⓐ}_{\text{bosonic}} = \underbrace{\left\{ \begin{matrix} \text{some combination} \\ \text{of } c_1^4, c_1^2 c_2, c_1 c_3, c_2^2 \end{matrix} \right\}}_{\substack{\mathrm{Vect}^{hol}(\mathbb{C}^3) \\ \text{part}}} + \# \overbrace{\int_X \mathrm{Tr}_{\mathfrak{sl}(2)} (A \partial A \partial A \partial A)}^{\mathfrak{sl}(2) \text{ KM part}}$$

[Faonte-Hennion-Kapranov]

+ mixed terms.

* Up to equivalence, we expect there to be a unique anomaly.

[For now, can show $H^7(E(3|6)) \longrightarrow H^7(E(3|6)_+)$ is injective.

Cor: The hol. twist of 6d (2,0) conformal sugra coupled to N M5 branes exists at the quantum level $\Leftrightarrow$ $N = 26$.

(See also Tseytlin's work on conformal sugra)]

* When $X = \mathbb{C}^3 \setminus 0$ get a central extension

$$\mathbb{C} \longrightarrow Vir_{Dol} \longrightarrow \mathcal{E}(\mathbb{C}^3 \setminus 0)$$

witt$_{3d}$

$sl(2) \otimes R\Gamma(\mathbb{C}^3 \setminus 0, \mathcal{O})$ [FHK]

$E(3|6)$ "positive part"

Will justify this name momentarily

$Vir_{Dol} \circlearrowright \{$ minimal BPS operators in 6d (2,0) theory $\}$.

* Conj for operators on a stack of $\underline{\underline{\text{two}}}$ fivebranes from twisted SUGRA: as a rep^n it is induced from $E(3|6) \subset Vir_{Dol}$. Character of this rep^n returns state of the art index computations (Kim^3 - Lee).

* AGT deformation: On $X = C \times \mathbb{C}^2$, C = Riem surface, there is a simplification of 6d (2,0) theory called the Ω-background. Mathematically, this is akin to taking $S^1 \curvearrowright \mathbb{C} \times \mathbb{C} \circlearrowleft S^1$ (derived) fixed points / equivariant cohomology.

* In the holomorphic twist: this process is much like "twisting"*

$$S_{\varepsilon_1, \varepsilon_2} \in \mathfrak{L}(3|6) \quad \text{MC element.}$$

$$\begin{array}{c} \mathrm{Obs} \\ \shortparallel \end{array} \quad \overset{\text{deform}}{\rightsquigarrow} \quad \left(\mathrm{Obs}, \; [S_{\varepsilon_1, \varepsilon_2}, -] \right)$$

fact. alg. of observables in the hol. twist.

generically localizes to a fact alg on $C \times \{0\} \subset C \times \mathbb{C}^2$.

* This leads to a mathematical statement [Nakajima, Grojnowski]

on $C = \mathbb{C}$

$$H^{\cdot}_{S^1 \times S^1}(\mathrm{Hilb}(\mathbb{C}^2))$$

$S^1 \times S^1$- equivariant de Rham cohomology

$\bigsqcup_n$ Hilbert scheme of n-pts on $\mathbb{C}^2$.

has the structure of a vertex algebra.

$\cong$ the chiral boson (Heisenberg) vertex algebra.

Thm: [Saberi - W] $\mathrm{Vir}_{\mathrm{Dol}} \xrightarrow[\text{"}\Omega\text{-backgrd"}]{[S,-]} \mathrm{Vir}$

Dolbeault Virasoro factorization algebra on $\mathbb{C} \times \mathbb{C}^2$

Virasoro vertex alg.

Dolbeault Virasoro: $\mathcal{E}(3|6)$ is defined for any 3-fold X.

When $X = S \times \mathbb{C}$, where S is complex surface, then AGT, Nakajima tells us there is a deformation (A-twist) of this that acts on $H^{\cdot}_{dR}(\mathrm{Hilb}(S))$. Before turning on this deformation, we expect

$$\mathcal{E}(S\times\mathbb{C}) \circlearrowright H^{\cdot}_{Dol}(\mathrm{Hilb}(S)).$$

of course, for S proper there is no difference (except for explicitly seeing the Hodge decomposition.) <u>But</u>, there is a generalization of this algebra which comes from M-theory on

Any line bundle

$$\mathbb{R} \times \mathrm{Tot}\begin{pmatrix} L\oplus L^{!} \\ \downarrow \\ S \end{pmatrix} \times \mathbb{C}$$

fivebrane.

where $L\otimes L^{!} = K_S$. Then, we expect that

$$\mathcal{E}_L(S\times\mathbb{C}) \circlearrowright H^{\cdot}_{\bar\partial}(\mathrm{Hilb}(S),\ \mathcal{L}_L)$$

[work in progress w/ Raghavendran] Tautological bundle from L.