Berkeley 21 Oct 2024

# 3d Mirror Symmetry is Mirror Symmetry

joint w/ Conan Leung

## Compare 2d & 3d mirror Symmetry

$Y$ CY mfd

⇝ 2d TQFTs

with objects/2d branes
$$\begin{tikzcd} E \arrow[d, "\text{family of 1d TQFTs (Vector bundles)}"] \\ L \underset{\text{Submfd}}{\subset} Y \end{tikzcd}$$

In A model, $L \subset Y$ is a Lagrangian submfd

$E/L$ is flat

In B model, $L \subset Y$ is a cpx submfd

$E/L$ is holomorphic

In 3d, a 3d brane would be a family of 2d TQFTs over a submfd.

## 2d Mirror Symmetry

| | | | | |
|---|---|---|---|---|
| Spaces | $\overset{A}{Y}$ | $\xleftrightarrow{\text{SYZ-transform}}$ | $\overset{B}{\check{Y}}$ | (or $(Y,D)$ $(\check{Y},W)$) |
| 2d branes | $\begin{matrix} E \\ \mid \\ L \subset Y \end{matrix}$ | $\xleftrightarrow{\text{Fourier-Mukai}}$ | $\begin{matrix} \mathcal{E} \\ \downarrow \\ Z \subset \check{Y} \end{matrix}$ | |
| moduli/deformation | symplectic structures | $\xleftrightarrow{\text{Mirror Map}}$ | complex structures | |

## 3d Mirror Symmetry:

| | | | | |
|---|---|---|---|---|
| Spaces (cpx symplectic) | $X$ | $\xleftrightarrow{\text{"gluing" by Lag. Corr.}}$ | $X^!$ | (pronounced $X$ – shriek) |
| 3d branes | $\begin{matrix} (Y,W) \\ \downarrow \\ C \underset{\text{cpx Lag}}{\subset} X \end{matrix}$ | $\xleftrightarrow{\text{SYZ-type transform}}$ | $\begin{matrix} (Z,J) \\ \downarrow \\ C \underset{\text{cpx Lag}}{\subset} X^! \end{matrix}$ | |
| deformations | Kähler parameters<br>Equiv. parameters | $\xleftrightarrow{\text{3d Mirror Map}}$ | Equiv. parameters<br>Kähler parameters | |

We propose:

I) 3d branes of $X$ and that of $X^!$ are related by an SYZ-type transform.

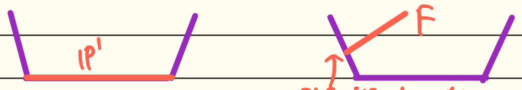
| Examples: | $X$ | $X^!$ | $Br(X)$ | $Br(X^!)$ | 2d interpretation |
|---|---|---|---|---|---|
| | $T^*[pt/T_{\mathbb{C}}]$ | $T^*\check{T}_{\mathbb{C}}$ | $T \curvearrowright Y \xrightarrow{\mu} \mathfrak{t}^\vee$ | $\check{Y} \to \check{T}_{\mathbb{C}}$ | Teleman Conj |
| | $T^*[\mathbb{C}/\mathbb{C}^\times]$ | $T^*\mathbb{C}$ | $(Y, D)$ | $(\check{Y}, f)$ | MS for log CY |
| | $T^*[\mathbb{C}^2_{1,-1}/\mathbb{C}^\times]$ | $T^*[\mathbb{C}_{1,-1}/\mathbb{C}^\times]$ | Tyurin deg. | fib / $\mathbb{P}^1$ | DHT Conj |

II) the exchange of symp & cpx str. in 2d MS induces the exchange of Kähler & equivariant parameters in 3d MS.

(3d Mirror map is Mirror map)

$$\begin{tikzcd} T^*\mathbb{P}^1 \arrow[r, leftrightarrow, "3d\ MS"] & T^*\mathbb{P}^1 \curvearrowleft \mathbb{C}^\times \\ \cup & \\ (\mathbb{P}^1, w) \arrow[r, leftrightarrow, "SYZ\text{-}trans"] & F \end{tikzcd}$$

<u>Example</u> :



precise location depends on $w$

Varying $w$ $\xleftrightarrow{SYZ}$ moving F using the $\mathbb{C}^\times$-action

III) Cpx Lag Corr. are 3d mirror to Cpx Lag. Corr.

If $X = \bigcup X_i$, $X_i \subset X$ open, there would be cpx Lag. Corr.

$$X_i^! \longrightarrow X^!$$

One may hope to build $X^!$ from these cpx Lagrangians.

Example:

$$\begin{tikzcd} & T^*\mathbb{P}^1 & \\ T^*\mathbb{C} \arrow[ur, phantom, "\subset" sloped] & & T^*\mathbb{C} \arrow[ul, phantom, "\supset" sloped] \\ & T^*\mathbb{C}^\times \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\supset" sloped] & \end{tikzcd} \qquad \begin{tikzcd} & T^*[\mathbb{C}^2_{1,-1}/\mathbb{C}^\times] & \\ T^*[\mathbb{C}_1/\mathbb{C}^\times] & & T^*[\mathbb{C}_{-1}/\mathbb{C}^\times] \\ & T^*[pt/\mathbb{C}^\times] & \end{tikzcd}$$

$\boxed{\text{no maps!}}$

should be related by Lag. Corr.

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} pt/\mathbb{C}^\times & \mathbb{C}_{-1}/\mathbb{C}^\times \arrow[l] \\ \mathbb{C}_1/\mathbb{C}^\times \arrow[u] & \mathbb{C}^2_{1,-1}/\mathbb{C}^\times \arrow[l] \arrow[u] \end{tikzcd}$$

pushout        pullback

Example (Teleman's gluing):

$$T^*pt/\mathbb{C}^\times \times T^*pt/\mathbb{C}^\times \longrightarrow\!\!\!\!\!\!\!+\ \ \ \ T^*pt/\mathbb{C}^\times \quad \text{induced by } \Delta : pt/\mathbb{C}^\times \to pt/\mathbb{C}^\times \times pt/\mathbb{C}^\times$$

branes level $\quad Y/\mathbb{C}^\times \times Y'/\mathbb{C}^\times \longrightarrow Y \times Y'/\mathbb{C}^\times$

substitutes $\quad Y' = V$ a $\mathbb{C}^\times$-representation

obtain $\quad T^*pt/\mathbb{C}^\times \longrightarrow\!\!\!\!\!\!\!+\ \ \ \ T^*pt/\mathbb{C}^\times$

mirror to $\quad T^*\mathbb{C}^\times \longrightarrow\!\!\!\!\!\!\!+\ \ \ \ T^*\mathbb{C}^\times$

$\uparrow$

this is Teleman's gluing map for $\mathcal{M}_C(\mathbb{C}^\times, V)$

Recall: SYZ: $\quad Y$ CY mfd

$\check{Y} = \{ (L, E) : L \subset Y \text{ Lag tori}, \ E/L \text{ unitary flat line bundle} \}$

More generally (Auroux)

$(Y, D)$ log CY

$\check{Y} = \{ (L, E) : L \subset Y \setminus D, \ E/L \text{ as before} \}$

$$W : \check{Y} \longrightarrow \mathbb{C}$$

$$(L, E) \longmapsto \sum_{\substack{\beta \in H_2(Y, L) \\ \beta \cdot D = 1}} n_\beta \, e^{-\int_\beta \omega} \, hol_{\partial\beta}(E)$$

E.g. $(Y, D) = (\mathbb{P}^1, \{0, \infty\})$, $(\check{Y}, W) = (\mathbb{C}^\times, z + \frac{1}{z})$ (or $z + q/z$, $q = \exp(-\text{Area}(\mathbb{P}^1))$)

more generally,

$Y$ Fano-toric with fan $\Sigma$

$(\check{Y}, W) =$ Hori-Vafa mirror

## I) 3d branes:

$\tilde{X}$ smooth cpx symplectic Variety, $G$ - cpt group

$G_{\mathbb{C}} \curvearrowright \hat{X} \xrightarrow{\mu_{\mathbb{C}}} \mathfrak{g}^{\vee}$ Hamiltonian action.

$X = [\tilde{X} /\!/ G_{\mathbb{C}}]$

**Def:** A 3d brane of $X$ is a the data of

- a $G_{\mathbb{C}}$-invariant cpx Lagrangian $\tilde{C} \subset \mu_{\mathbb{C}}^{-1}(0) \subset \tilde{X}$.
- a Hamiltonian $G$-Kähler mfd $Y$
- a $G$-equivariant holomorphic map $Y \to \tilde{C}$

We denote it as $[Y /\!/ G] \to C := [\tilde{C} / G_{\mathbb{C}}]$

or $\underline{C} = (C, [Y /\!/ G])$.

**Example:**

1) Any Hamiltonian $S^1$-Kähler mfd $Y$

gives a 3d brane $[Y /\!/ S^1] \to [pt/\mathbb{C}^{\times}]$ of $T^*[pt/\mathbb{C}^{\times}] = [pt /\!/ \mathbb{C}^{\times}]$

2) Any Kähler mfd $Y$ with a holomorphic map $\pi : Y \to \mathbb{C}^{\times}$

gives a 3d brane $Y \to \mathbb{C}^{\times}$ of $T^*\mathbb{C}^{\times}$

Teleman's 2014 ICM address:

$$\begin{tikzcd} S^1 \curvearrowright Y \arrow[r, rightsquigarrow] & \check{Y} \arrow[d, "\check{\pi}"] \\ & \mathbb{C}^{\times} \end{tikzcd}$$

(s.t. $Y /\!/_{\lambda} S^1$ is mirror to $\check{\pi}^{-1}(e^{-\lambda})$)

It says 3d brane in $T^*[pt/\mathbb{C}^{\times}]$ induces a 3d brane in $T^*\mathbb{C}^{\times}$.

(the other statement is related to Hom, we skip today)

# 3d Mirror Symmetry for hypertoric

$$1 \to \mathbb{C}^{\times l-n} \to \mathbb{C}^{\times l} \to \mathbb{C}^{\times n} \to 1$$

$$1 \leftarrow \check{\mathbb{C}}^{\times l-n} \leftarrow \check{\mathbb{C}}^{\times l} \leftarrow \check{\mathbb{C}}^{\times n} \leftarrow 1$$

dual exact sequences.

$$X = T^*\mathbb{C} /\!\!/ \mathbb{C}^{\times l-n} \qquad X^! = T^*\check{\mathbb{C}}^l /\!\!/ \mathbb{C}^{\times n}$$

For 2d mirror symmetry considerations, we consider

**Def.** Let $\mathbb{P}_\Sigma$ be a smooth toric variety with dense torus $T^n_{\mathbb{C}} \cong (\mathbb{C}^\times)^n$, and fan $\Sigma$.

Each $u \in \Sigma(1)$ is a homomorphism $u : \mathbb{C}^\times \to T^n_{\mathbb{C}}$,

or equivalently, a dual homomorphism $\check{u} : \check{T}^n_{\mathbb{C}} \to \mathbb{C}^\times$

We define $\mathbb{P}^!_\Sigma = \left[\mathbb{C}^l / \check{T}^n_{\mathbb{C}}\right]$, where $\Sigma(1) = \{u_1, \dots, u_l\}$

$\check{T}^n_{\mathbb{C}} \curvearrowright \mathbb{C}^l$ through

$$\check{T}^n_{\mathbb{C}} \xrightarrow{\begin{pmatrix}\check{u}_1 \\ \vdots \\ \check{u}_l\end{pmatrix}} \mathbb{C}^{\times n} \curvearrowright \mathbb{C}^n$$

more generally, if $T_{\mathbb{C}}$ is another cpx torus

$p : T_{\mathbb{C}} \to T^n_{\mathbb{C}}$ is a homomorphism

$C = \left[\mathbb{P}_\Sigma / T_{\mathbb{C}}\right]$

We define $C^! = \left[\mathbb{C}^l \times \check{T}_{\mathbb{C}} / \check{T}^n_{\mathbb{C}}\right]$,

where $\check{T}^n_{\mathbb{C}} \curvearrowright \check{T}_{\mathbb{C}}$ through $\check{p} : \check{T}^n_{\mathbb{C}} \to \check{T}_{\mathbb{C}}$.

**Examples:**

1) If $T_{\mathbb{C}} = 1$, $\mathbb{P}_\Sigma$ comes from a GIT quotient $\mathbb{C}^\ell /\!/ \mathbb{C}^{\times \ell - n}$

Consider the sequence as before

$$1 \to \mathbb{C}^{\times \ell - n} \to \mathbb{C}^{\times \ell} \to \mathbb{C}^{\times n} \to 1$$

$$1 \leftarrow \check{\mathbb{C}}^{\times \ell - n} \leftarrow \check{\mathbb{C}}^{\times \ell} \leftarrow \check{\mathbb{C}}^{\times n} \leftarrow 1$$

then $C^! = [\check{\mathbb{C}}^\ell / \check{\mathbb{C}}^{\times n}]$

2) If $\mathbb{P}_\Sigma = \mathbb{C}^\ell$, $T_{\mathbb{C}} = \mathbb{C}^{\times \ell - n}$,

then $C = [\mathbb{C}^\ell / \mathbb{C}^{\times \ell - n}]$, $C^! = [\check{\mathbb{C}}^\ell \times \check{\mathbb{C}}^{\times \ell - n} / \check{\mathbb{C}}^{\times \ell}] \cong [\check{\mathbb{C}}^\ell / \check{\mathbb{C}}^{\times n}]$

3) If $\mathbb{P}_\Sigma = pt$, $T_{\mathbb{C}} = \mathbb{C}^\times$

then $C = [pt/\mathbb{C}^\times]$, $C^! = \mathbb{C}^\times$.

We consider 3d Mirror pairs

$$X = T^*C, \quad X^! = T^*C^!$$

Roughly, we give

$$\left\{\begin{array}{c}\text{3d branes of} \\ T^*C \text{ supported} \\ \text{on } C\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{3d branes with potentials} \\ \text{of } T^*C^!\text{, supported} \\ \text{on } C^!\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{3d branes} \\ \text{of } T^*C^!\end{array}\right\}$$

**Description:**

**Case 1:** (Also due to Teleman) $C = [pt/\mathbb{C}^\times]$, $C^! = \mathbb{C}^\times$

$S^1 \curvearrowright Y$ $\quad ([Y/\!/T] \to [Y/\!/T_{\mathbb{C}}]$

$(Y, D)$ Log CY, $D$ $T$-inv.

$\check{Y} = \{(L, E) : L \subset Y \setminus D$ is an $S^1$-inv Lag, $E/L$ flat unitary$\}$

$$\check{x} : \check{Y} \longrightarrow \mathbb{C}^\times = \mathbb{R} \times S^1$$
$$(L, E) \longmapsto (\mu(L), hol_{S^1}(E))$$

– $C = [pt/T_{\mathbb{C}}]$, $C^! = \check{T}_{\mathbb{C}}$ similar

**Case 2 :** $C = \mathbb{C}^\times$ , $C^! = [pt/\mathbb{C}^\times]$

$Y \xrightarrow{\pi} \mathbb{C}^\times$ $(Y,D)$ log CY

$$\begin{tikzcd}
\{\text{unitary flat bundle on } \mathbb{C}^\times\}/\cong \arrow[r, bend left, "\text{tensor product}"'] \arrow[d, phantom, "\cong" rotate=90] & \check{Y} = \{(L,E)\} \\
S^1 &
\end{tikzcd}$$

Explicitly, for $z \in S^1$, let $\mathcal{L}_z$ be a unitary flat line bundle on $\mathbb{C}^\times$ with $\mathrm{mon}(\mathcal{L}_z) = z$

then
$$S^1 \times \check{Y} \to \check{Y}$$
$$\big(z\ ,\ (L,E)\big) \mapsto \big(L, E \otimes \pi^* \mathcal{L}_z|_L\big)$$

Moreover, if we impose $L \subset Y \setminus D$ to be special Lagrangians w.r.t. a meromorphic form $\Omega_Y$ with $\mathrm{div}(\Omega_Y) + D \equiv 0$.

then
$$\check{Y} \to \mathbb{R}$$
$$(L,E) \mapsto \int_L \log|\pi| \cdot \mathrm{Im}\,\Omega_Y$$
is a moment map.

**Example :**
$$\begin{tikzcd}
\mathbb{C}^{\times n} \arrow[d, "\prod x^i"] & \longleftrightarrow & \check{\mathbb{C}}^{\times n} \arrow[r, phantom, "\curvearrowleft"] & \mathbb{C}^\times \text{ diagonal action} \\
\mathbb{C}^\times & & &
\end{tikzcd}$$

**Case 3 :** $C = \mathbb{P}_\Sigma$ $C^! = \left[\mathbb{C}^\ell / T^{\vee n}_{\mathbb{C}}\right]$

$(Y,D)$ log CY

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[r, phantom, "\supset"] & \mathring{Y} \arrow[d] \\
\bigcup_{i=1}^{\ell} H_i \arrow[r, phantom, "\subset"] & \mathbb{P}_\Sigma \arrow[r, phantom, "\supset"] & T^n_{\mathbb{C}}
\end{tikzcd}$$

Assume $D = D_0 + D_1 + \cdots + D_\ell$ , with $D_i = \pi^{-1}(H_i)$ for $i > 0$

and no components of $D_0$ is contained in $\bigcup_{i=1}^{\ell} D_i$ .

$$\check{Y} = \{(L,E) : L \subset \underset{\substack{\shortparallel \\ \mathring{Y} \setminus D_0}}{Y \setminus D}\ ,\ E/L \text{ flat unitary}\}/\cong$$

Case 2: $\exists\ \check{T}^n \curvearrowright \check{Y}$

**Lemma** (C., Leung) Let $\check{Y} \xrightarrow{F_i} \mathbb{C}$

$$F_i(L,E) = \sum_{\substack{\beta \in H_2(Y,L) \\ \beta \cdot D = \beta \cdot D_i = 1}} n_\beta \, e^{-\int_\beta \omega} \, hol_{\partial\beta}(E)$$

then i) $F_0$ is $\check{T}^n$-invariant.

ii) for $i>0$, $F_i : \check{Y} \to \mathbb{C}$ is $\check{T}^n$-equivariant when $\check{T}^n$ acts on $\mathbb{C}$ via $\check{T}^n \subset \check{T}^n_{\mathbb{C}} \xrightarrow{\check{u}_i} \mathbb{C}^\times$.

As a corollary

$$F = (F_1, \dots, F_\ell) : \check{Y} \longrightarrow \mathbb{C}^\ell$$

is $\check{T}^n$-equivariant

providing a 3d brane (with potential)

$$\begin{tikzcd} \left[ (\check{Y}, F_0) /\!\!/ \check{T}^n \right] \arrow[d] \\ \left[ \mathbb{C}^\ell / \check{T}^n_{\mathbb{C}} \right] = \mathcal{C}^! \end{tikzcd}$$

**Example:**

$$\begin{tikzcd} \mathbb{C}^n \arrow[d, "\Pi x_i"] & & \left[\mathbb{C}^{\times n} / \mathbb{C}^\times\right] \arrow[d, "\Sigma x_i"] \\ \mathbb{C} & \text{is 3d mirror to} & \left[\mathbb{C} / \mathbb{C}^\times\right] \end{tikzcd}$$

**Case 4:** (Combination of Steps 1, 2, 3) $\mathcal{C} = \left[\mathbb{P}_\Sigma / \check{T}_{\mathbb{C}}\right]$, $\mathcal{C}^! = \left[\mathbb{C}^\ell \times \check{T}_{\mathbb{C}} / \check{T}^n_{\mathbb{C}}\right]$

$Y \longrightarrow \mathbb{P}_\Sigma$ T-equiv.

Step 1 :

$$\begin{tikzcd} \check{\pi} : \check{Y} \arrow[r] & \check{T}_{\mathbb{C}} \\ (\text{step2}) \circlearrowright \check{T}^n & \check{T}^n_{\mathbb{C}} \arrow[u, "\check{\rho}"'] \end{tikzcd}$$

**Lemma** (C., Leung) $\check{\pi}$ is $\check{T}^n$-equivariant.

Combining with step 3, we obtain

$$\begin{tikzcd} \left[\check{Y} /\!\!/ \check{T}^n\right] \arrow[d, "{(F, \check{\pi})}"] \\ \left[\mathbb{C}^\ell \times \check{T}_{\mathbb{C}} / \check{T}^n_{\mathbb{C}}\right]. \end{tikzcd}$$

**Thm** (C., Leung) Assume convergence

$$\begin{tikzcd} {[(Y,D)/\!\!/T]} \arrow[dd] & & {[(\check{Y},F_0)/\!\!/\check{T}^n]} \arrow[dd] \\ & \rightsquigarrow & \\ {[\mathbb{P}_\Sigma/T_\mathbb{C}]} & & {[\mathbb{C}^\ell \times \check{T}_\mathbb{C}/\check{T}^n_\mathbb{C}]} \end{tikzcd}$$

## II. Cpx Lagrangian Corr. and Functoriality of the transform

**Example:** By our construction, we have

$$\begin{tikzcd} Y \arrow[d] & \supset & \mathring{Y} \arrow[d] & \text{is 3d mirro to} & {[\check{Y}/\!\!/S^1]} \arrow[d] & = & {[\check{Y}/\!\!/S^1]} \arrow[d] \\ \mathbb{C} & \supset & \mathbb{C}^\times & & {[\mathbb{C}/\mathbb{C}^\times]} \arrow[rr] & & {[pt/\mathbb{C}^\times]} \end{tikzcd}$$

i.e. $\exists$ commutative diagram

$$\begin{tikzcd} Br(T^*\mathbb{C}) \arrow[r, "3d"] \arrow[d, "restriction"] & Br(T^*[\mathbb{C}/\mathbb{C}^\times]) \arrow[d, "pushforward"] \\ Br(T^*\mathbb{C}^\times) \arrow[r, "3d"] & Br(T^*[pt/\mathbb{C}^\times]) \end{tikzcd}$$

← should be viewed as Lag. Corr.

Similarly, one might expect

$$\begin{tikzcd} Y \arrow[d] & \supset & D \arrow[d] & \text{is 3d mirror to} & {[\check{Y}/\!\!/S^1]} \arrow[d] & \supset & \check{Y}/\!\!/S^1 \arrow[d] \\ \mathbb{C} & \ni & 0 & & {[\mathbb{C}/\mathbb{C}^\times]} & \supset & \mathbb{C}^\times/\mathbb{C}^\times \simeq pt \end{tikzcd}$$

**Example:**

$$\begin{tikzcd} \mathbb{C}^n \arrow[d, "\Pi x_i"] & \supset & \{\Pi x_i = 0\} \arrow[d] & & \mathbb{C}^{\times n}/\mathbb{C}^\times \arrow[d, "\Sigma x_i"] & \supset & \{\Sigma x_i \neq 0\}/\mathbb{C}^\times \approxeq (n-1)\text{-dim pair of pants.} \arrow[d] \\ \mathbb{C} & \ni & 0 & & {[\mathbb{C}/\mathbb{C}^\times]} & \supset & pt \end{tikzcd}$$

$\exists$ Commutative diagram

$$\begin{tikzcd} Br(T^*\mathbb{C}) \arrow[r, "3d"] \arrow[d, "\Phi"'] & Br(T^*[\mathbb{C}/\mathbb{C}^\times]) \arrow[d, "\Phi^!"] \\ Br(pt) \arrow[r, "3d"] & Br(pt) \end{tikzcd}$$

$\Phi$ should be viewed as induced by the Lag. Corr.

$$\mathbb{C} \times pt \subset T^*\mathbb{C} \times pt$$

$\Phi^!$ should be viewed as induced by the Lag. Corr.

$$\mathbb{C}^\times/\mathbb{C}^\times \times pt \subset T^*[\mathbb{C}/\mathbb{C}^\times] \times pt$$

By our 3d transform

$$\begin{tikzcd}
\mathbb{C} \arrow[d] & \text{is 3d mirror to} & \mathbb{C}^\times/\mathbb{C}^\times \arrow[d] \\
\mathbb{C} = \mathbb{C} \times pt \subset T^*\mathbb{C} \times pt & & {[\mathbb{C}/\mathbb{C}^\times]} = [\mathbb{C}/\mathbb{C}^\times] \times pt \subset T^*[\mathbb{C}/\mathbb{C}^\times] \times pt.
\end{tikzcd}$$

3d mirror map is mirror map

**Example:**

$$\begin{tikzcd} Y = \mathbb{P}^n \arrow[d] & & \mathbb{C}^{\times n}/\mathbb{C}^{\times n} \arrow[d, "z_1+\cdots+z_n+\frac{q}{z_1\cdots z_n}"] \\ C = \mathbb{P}^n \subset T^*\mathbb{P}^n \arrow[r, phantom, "\text{is 3d mirror to}"] & & {[\mathbb{C}^{n+1}/\mathbb{C}^{\times n}]} \end{tikzcd}$$

$$\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+1}, \quad wt = (0,\cdots,0,1)$$

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[d] \\ \curvearrowright \; {[\mathbb{C}^{n+1}/\mathbb{C}^\times]} \end{tikzcd}$$

deform $\begin{tikzcd} Y \arrow[d] \\ C \end{tikzcd}$ by varing kähler class of $Y$ via pullbrek of kähler class of $C$ (or $T^*\mathbb{P}^n$)

$\overset{\text{3d mirror}}{\sim}$ deform $\begin{tikzcd} \check{Y}/\mathbb{C}^{\times n+1} \arrow[d] \\ C^! \end{tikzcd}$ by a Hamiltonian $\mathbb{C}^\times$-action

**Example:**

$$\begin{tikzcd} Y \arrow[d, "\pi"] & & \check{Y} \arrow[d, "(F_0, F_\infty)"] & /\mathbb{C}^\times \\ \mathbb{P}^1 & & \mathbb{C}^2_{1,-1} & /\mathbb{C}^\times \end{tikzcd}$$

where $D_0 = \pi^{-1}(0)$, $D_\infty = \pi^{-1}(\infty)$

$$F_0(L,E) = \sum_{\substack{\beta\cap D_0 = 1 \\ \beta\cap D_\infty = 0}} n_\beta \, e^{-\int_\beta \omega_Y} hol_{\partial\beta}(E)$$

$$F_\infty(L,E) = \sum_{\substack{\beta\cap D_0 = 0 \\ \beta\cap D_\infty = 1}} n_\beta \, e^{-\int_\beta \omega_Y} hol_{\partial\beta}(E)$$

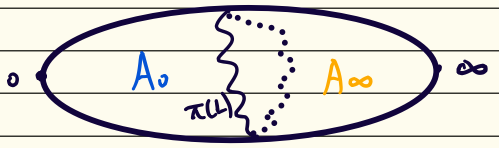
If $\omega \in H^2(T^*\mathbb{P}^2)$

$$\begin{tikzcd} (Y, \omega_t = \omega_Y + t\pi^*\omega) \arrow[d] & \rightsquigarrow & \check{Y} \arrow[d, "(F_0^t, F_\infty^t)"] & /\mathbb{C}^\times \\ \mathbb{P}^1 & & \mathbb{C}^2_{1,-1} & /\mathbb{C}^\times \end{tikzcd}$$

where $F_0^t(L,E) = \sum_{\substack{\beta\cap D_0 = 1 \\ \beta\cap D_1 = 0}} n_\beta \, e^{-\int_\beta \omega_Y + t\pi^*\omega} hol_{\partial\beta}(E)$

$$= e^{-\int_{A_0} \omega} F_0(L,E)$$

where



Similarly, $F_\infty^t(L,E) = e^{-\int_{A_\infty} \omega} F_\infty(L,E)$

The map

$$\Phi : H^2(T^*\mathbb{P}^1) \longrightarrow t^!_{\mathbb{C}} = \mathbb{C}^2_{1,-1}/\mathbb{C}$$
$$\omega \longmapsto \left(\int_{A_0} \omega, \int_{A_\infty} \omega\right)$$

is independent of $L$.

Moreover, deforming $Y \to \mathbb{P}^1$ by $\pi^* \omega$

is 3d mirror to

deforming $Y^\vee \to [\mathbb{C}^2_{1,-1}/\mathbb{C}^\times]$ by $\exp(-\Phi(\omega))$

An easy consequence

<u>Cor</u>. If $\omega$ is Kähler, then $\underline{C}^!$ is attracting w.r.t. $\Phi(\omega)$

<u>proof</u>: because $\int_{A_i} \omega > 0$ for $\omega$ käher.

# Product structure on $Br(T^*[pt/T_{\mathbb{C}}])$

For simplicity, take $T_{\mathbb{C}} = \mathbb{C}^\times$.

∃ product structure

$$Br(T^*[pt/\mathbb{C}^\times]) \times Br(T^*[pt/\mathbb{C}^\times]) \xrightarrow{\mathcal{L}_o} Br(T^*[pt/\mathbb{C}^\times])$$

$$(S^1 \curvearrowright Y_1), (S^1 \curvearrowright Y^2) \longmapsto S^1 \curvearrowright Y_1 \times Y_2$$

It is induced by the Lagrangian correspondence

$$N^*\left(pt/\mathbb{C}^\times \xrightarrow{\Delta} pt/\mathbb{C}^\times \times pt/\mathbb{C}^\times \times pt/\mathbb{C}^\times\right) \subset \left(T^*[pt/\mathbb{C}^\times]\right)^3$$

$$\begin{tikzcd}
\mathbb{C}^{\times 2}/\mathbb{C}^{\times 3} = pt/\mathbb{C}^\times \arrow[d] & \text{is 3d mirror to} & \mathbb{C}^{\times 2} \arrow[d, "{(a,b,ab)}"] & \\
(pt/\mathbb{C}^\times)^3 & & \mathbb{C}^{\times 3} & \subset (T^*\mathbb{C}^\times)^3
\end{tikzcd}$$

whose conormal gives a symplectic groupoid structure on $T^*\mathbb{C}^\times$

$$\mathcal{L}_m = \{(z_1, h, z_2, h, z_1 z_2, h) \in (T^*\mathbb{C}^\times)^3 = (\mathbb{C}^\times \times \mathbb{C})^3\}$$

In what sense $\mathcal{L}_o$ and $\mathcal{L}_m$ are mirror Lag. Corr.?

Recall

$$\left\{\begin{array}{c}\text{3d branes of } T^*pt/\mathbb{C}^\times \\ \text{supported on } pt/\mathbb{C}^\times\end{array}\right\} \xrightarrow[\text{SYZ-type}]{①} \left\{\begin{array}{c}\text{3d branes with potentials of } T^*\mathbb{C}^\times \\ \text{supported on } \mathbb{C}^\times\end{array}\right\}$$

$$\xrightarrow{②} \{\text{3d branes on } T^*\mathbb{C}^\times\}$$

We describe ② now:

$$\begin{tikzcd}
\text{Suppose} & (\check{Y}, F) \arrow[d, "\pi"] \\
& \mathbb{C}^\times
\end{tikzcd}$$

then Conormal to the graph $\Gamma_\pi$ give a Lag. Corr.

$$T^*Y \xrightarrow{\mathcal{L}_\pi} T^*\mathbb{C}^\times$$

$\Gamma_{dF} \subset T^*Y$ is a Lagrangian

② sends $$\begin{tikzcd}(Y,F) \arrow[d, "\pi"] \\ \mathbb{C}^\times\end{tikzcd}$$ to $\mathcal{L}_\pi(\Gamma_{dF})$

<u>Examples:</u> $\mathbb{C}^\times \curvearrowright \mathbb{C}^n$ diagonal $\rightsquigarrow$ $$\begin{tikzcd}\mathbb{C}^{\times n}, F = \Sigma x_i \arrow[d, "\pi = \Pi x_i"] \\ \mathbb{C}^\times\end{tikzcd}$$

$$\Gamma_{dF} = \{ (x_1, \dots, x_n, x_1, \dots, x_n) \in \mathbb{C}^{\times n} \times \mathbb{C}^n \}$$

$$b_n := \mathcal{L}_\pi(\Gamma_{dF}) = \{ (z,h) \in \mathbb{C}^\times \times \mathbb{C} : z = h^n \}$$

In particular. $\mathcal{L}_m(b_a, b_b) = b_{a+b}$

# Gluing Construction of 3d mirror to gauge theory

**Example** $X = [T^* \mathbb{C} /\!/ \mathbb{C}^\times]$

$$\begin{tikzcd}
Br(X) \arrow[r, "3d"] \arrow[d, shift right, "F_i"'] & Br(X^!) \arrow[d, shift right, "F_i^!"'] \\
Br(T^*[pt/\mathbb{C}^\times]) \arrow[r] \arrow[u, shift right, "G_i"'] & Br(T^*\mathbb{C}^\times) \arrow[u, shift right, "G_i^!"']
\end{tikzcd}$$

$$F_2^{-1} \circ F_1 : Br(T^*[pt/\mathbb{C}^\times]) \to Br(T^*[pt/\mathbb{C}^\times])$$

$$S^1 \curvearrowright Y \longmapsto S^1 \curvearrowright Y \times \mathbb{C}$$

$$F_2^{!} \circ F_1^{\overset{!}{-}} : Br(T^*\mathbb{C}^\times) \longrightarrow Br(T^*\mathbb{C}^\times)$$

$$C \longmapsto \mathcal{L}m(C, \mathcal{O}_1)$$

induced by the birational map

$$T^*\mathbb{C}^\times \longrightarrow T^*\mathbb{C}^\times$$

$$(z, h) \longmapsto (zh, h)$$