WIDE STABILITY

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1. Wide ultrafilters

1.1. **Quotient algebras and their ultrafilters.** Let B be a Boolean algebra and let $I \subset B$ be an ideal. Then I uniquely determines a congruence $\equiv$ on B given by $x \equiv y \iff x \Delta y \in I$. Accordingly, we denote the $\equiv$-class of an element $x \in B$ by x/I and the set of $\equiv$-classes of elements of B by B/I. A subset $A \subset B$ is called *I-saturated* if it is saturated with respect to the associated congruence, i.e. $x \in A$ and $x \Delta y \in I$ implies $y \in A$.

The Boolean operations on B descend to well-defined operations on the quotient B/I in the obvious way: $x/I \vee y/I := (x \vee y)/I$, $(x/I)' := x'/I$, etc. Moreover, these operations equip B/I with the structure of a Boolean algebra. The quotient map $\pi : B \to B/I$ is thus an epimorphism in the category of Boolean algebras with Boolean algebra homomorphisms.

A consequence of the Stone representation theorem is that π induces a monomorphism $\pi^* : S(B/I) \to S(B)$ in the category of Stone (i.e., compact Hausdorff totally disconnected) spaces; concretely, π^* is given by

$$\pi^*(\mathcal{U}) = \pi^{-1}(\mathcal{U}).$$

As continuous injections from compact to Hausdorff spaces are automatically embeddings, it follows that $S(B/I)$ is homeomorphic to a subspace of $S(B)$. The image of $S(B/I)$ under π^* has a natural description:

Definition 1.1. An ultrafilter $\mathcal{U} \in S(B)$ is *I-wide* if it avoids the ideal I. Let $S^I(B) = \{\mathcal{U} \in S(B) : \mathcal{U} \cap I = \emptyset\}$ denote the closed subspace of I-wide ultrafilters on B.

Observation 1.2. Every $\mathcal{U} \in S^I(B)$ is I-saturated.

Proof. Let $\mathcal{U} \in S(B)$ and assume there exist $x, y \in B$ such that $x \in \mathcal{U}$, $y \equiv x$, but $y \notin U$. Then $y' \in \mathcal{U}$ by maximality of $\mathcal{U}$, whence $x \wedge y' \in \mathcal{U}$. On the other hand $x \equiv y$ implies

$$x \wedge y' \equiv y \wedge y' = \bot,$$

i.e. $x \wedge y' \in I$. Thus $\mathcal{U}$ is not I-wide. □

Proposition 1.3. *π^* maps $S(B/I)$ homeomorphically onto $S^I(B)$.*

Proof. We have already established that π^* is an embedding; it remains to show the image of $S(B/I)$ under π^* is $S^I(B)$.

First, let $\mathcal{U} \in S(B/I)$. If there is an $x \in \pi^{-1}(\mathcal{U}) \cap I$, then $\bot/I = \pi(x) \in \mathcal{U}$, which is impossible because $\mathcal{U}$ is an ultrafilter on B/I. So we have $\pi^*(S(B/I)) \subset S^I(B)$.

Now, let $\mathcal{V} \in S^I(B)$. We want to show there exists $\mathcal{U} \in S(B/I)$ such that $\pi^*(\mathcal{U}) = \mathcal{V}$. The natural candidate is $\mathcal{U} := \pi(\mathcal{V})$.

First, we show $\pi(\mathcal{V}) \in S(B/I)$ for $\mathcal{V} \in S^I(B)$. Clearly $\perp/I \notin \pi(\mathcal{V})$ since $\mathcal{V}$ avoids I. Suppose $x/I, y/I \in \pi(\mathcal{V})$. As $\mathcal{V}$ is wide, it is in particular saturated; it follows that $x, y \in \mathcal{V}$. So $x/I \wedge y/I = \pi(x \wedge y) \in \pi(\mathcal{V})$. Now suppose $x/I \in \pi(\mathcal{V})$, $y/I \notin \pi(\mathcal{V})$, but $y \wedge y' \equiv \perp$. Using saturation and maximality, respectively, we see $x, y' \in \mathcal{V}$. Thus $x \wedge y' \in \mathcal{V}$ yet $x \wedge y' \in I$, contradicting the assumption that $\mathcal{V}$ is wide. Finally, the maximality of $\mathcal{V}$ implies that of $\pi(\mathcal{V})$.

Next, we prove $\pi^*(\pi(\mathcal{V})) = \mathcal{V}$ for $\mathcal{V} \in S^I(B)$. Clearly $\mathcal{V} \subset \pi^{-1}\pi(\mathcal{V})$. Let $x \in \pi^{-1}\pi(\mathcal{V})$, so that $x \equiv y$ for some $y \in \mathcal{V}$. Since wide ultrafilters are saturated, it follows, $x \in \mathcal{V}$. □

1.2. **The I-finite meet property.**

Definition 1.4. A subset $F \subset B$ has the *I-finite meet property* (I-FMP) if $\bigwedge X_0 \notin I$ for every finite $X_0 \subset X$. X has the *finite meet property* (FMP) if it has the $\{\perp\}$-FMP.

Lemma 1.5. *The set of I-FMP, I-saturated subsets of B bijects with the set of FMP subsets of B/I via the maps $X \mapsto \pi(X)$ and $Y \mapsto \pi^{-1}(Y)$.*

Proof. First, we show that these maps are well-defined. Suppose $Y \subset B/I$ has the finite meet property and put $X = \pi^{-1}(Y)$. Clearly X is saturated with respect to I. Moreover, it is I-FMP: if $\bigwedge X_0 \in I$ for some finite $X_0 \subset X$, then $\perp/I = \bigwedge_{x \in X_0} \pi(x)$, which is impossible if Y is FMP. Now suppose $X \subset B$ is I-FMP and I-saturated and put $Y = \pi(X)$. Let $Y_0 \subset Y$ be finite, so that $Y_0 = \{\pi(x) : x \in X_0\}$ for some finite $X_0 \subset X$. As X is I-FMP, we have $\bigwedge X_0 \notin I$. It follows, $\bigwedge Y_0 = \pi\left(\bigwedge X_0\right) \neq \perp/I$.

Next, we show the correspondence is bijective. For FMP $Y \subset B/I$, $\pi^{-1}\pi(Y) = Y$ is immediate because π is surjective. On the other hand, suppose $X \subset B$ is I-FMP and I-saturated. $X \subset \pi^{-1}\pi(X)$ is automatic, so it remains to show the reverse inclusion. Let $x \in \pi^{-1}\pi(X)$, so that $x \equiv y$ for some $y \in X$. As X is I-saturated, we conclude $x \in X$, and we are done. □

Lemma 1.6. *For $X \subset B$ let $\hat{X} = \bigcup\{x/I : x \in X\}$ be its I-saturation. If X is I-FMP, then so is $\hat{X}$.*

Proof. Let $\hat{X}_0 \subset \hat{X}$ be finite. By definition, for each $x \in \hat{X}_0$ there is $x^* \in X$ such that $x \equiv x^*$. Let $X_0 = \{x^* : x \in \hat{X}_0\}$. Since X is I-FMP, we must have $\bigwedge X_0 \notin I$. But $\bigwedge \hat{X}_0 \equiv \bigwedge X_0$ and I is closed under $\equiv$, so the desired conclusion follows. □

Corollary 1.7. *Any I-FMP subset of B extends to an I-wide ultrafilter on B.*

Proof. Let $X \subset B$ be I-FMP. By the preceding lemma, its I-saturation $\hat{X}$ is also I-FMP. In view of Lemma 2.5, $\hat{X}$ thus determines an FMP subset $\hat{Y} = \pi\left(\hat{X}\right)$ of B/I. Its upward closure

$$\left\{y \in B/I : z \leq y \text{ for some } z \in \hat{Y}\right\}$$

is a filter on B/I containing $\hat{Y}$, which thus extends to an ultrafilter $\mathcal{V}$ on B/I by Zorn's lemma. But $\mathcal{V}$ in turn corresponds to a wide ultrafilter $\mathcal{U} = \pi^{-1}(\mathcal{V})$ on B by Proposition 2.3, and moreover we have

$$X \subset \hat{X} = \pi^{-1}\pi\left(\hat{X}\right) \subset \pi^{-1}(\mathcal{V}) = \mathcal{U}.$$

This completes the proof. □

1.3. **Lindenbaum–Tarski algebras and model-theoretic setup.** Fix a first-order language $\mathcal{L}$ and a complete $\mathcal{L}$-theory T. Following standard model-theoretic convention, we will assume the existence of a saturated model $\mathbb{U} \models T$ of some uncountable inaccessible cardinality. A subset $A \subset \mathbb{U}$ is called small if $|A| < |\mathbb{U}|$. If the reader does not wish to impose additional set-theoretic hypotheses, it is enough to consider a κ-saturated and strongly κ-homogeneous $\mathbb{U} \models T$ for a "sufficiently large" uncountable regular κ. In this case "small" means of cardinality less than κ.

Let $A \subset \mathbb{U}$. $\mathcal{L}(A)$ refers to the expansion of $\mathcal{L}$ by fresh constants c_a for $a \in A$. $\mathbb{U}$ naturally expands to an $\mathcal{L}(A)$-structure which interprets each c_a as a; thus we abuse notation and call this structure $\mathbb{U}$ as well. $T(A)$ denotes the complete $\mathcal{L}(A)$-theory of $\mathbb{U}$.

Fix a tuple x of free variables. $\mathcal{L}_x(A)$ denotes the set of $\mathcal{L}(A)$-formulas whose free variables are among those in x. The Lindenbaum–Tarski algebra $L_x(A)$ is obtained as the quotient of $\mathcal{L}_x(A)$ by the equivalence relation

$$\psi(x) \sim \theta(x) \iff \mathbb{U} \models \forall x.\psi(x) \leftrightarrow \theta(x),$$

and it forms a Boolean algebra when the Boolean operations are interpreted as the corresponding logical connectives, i.e. $\psi(x)/\sim \vee \theta(x)/\sim := (\psi(x) \vee \theta(x))/\sim$, $(\psi(x)/\sim)' := \neg\psi(x)/\sim$, etc. Abusing notation, we identify $\sim$-classes $\psi(x)/\sim$ with their representatives $\psi(x)$ unless the circumstances demand more care.

Fix a partitioned formula $\varphi = \varphi(x, y)$. $L_\varphi(A)$ denotes the Boolean subalgebra of $L_x(A)$ generated by $\{\varphi(x, a) : a \in A^{|y|}\}$. Elements of $L_\varphi(A)$ are called *φ-formulas over A*.

$S_x(A)$ denotes the Stone space of the algebra $L_x(A)$; likewise $S_\varphi(A)$ denotes the Stone space of the algebra $L_\varphi(A)$.

Let $I \subset L_x(\mathbb{U})$ be an ideal. We define $I(A) := I \cap L_x(A)$ and $I_\varphi(A) = I \cap L_\varphi(A)$; these are ideals of $L_x(A)$ and $L_\varphi(A)$, respectively. A formula $\psi(x) \in L_x(\mathbb{U})$ is *thin* (with respect to I) if $\psi(x) \in I$, *wide* if $\psi(x) \notin I$,

co-wide if $\neg\psi(x)$ is wide, and *full* if $\neg\psi(x)$ is thin. A *complete* type $p \in S_x(A)$ or $S_\varphi(A)$ is *wide* if $p \cap I = \emptyset$, and *thin* otherwise; a partial type π is *wide* if $\bigwedge \pi_0$ is wide for every finite $\pi_0 \subset \pi$, and *thin* otherwise.

1.4. **Ideals of low descriptive complexity; extensions to superalgebras.** Let $A \subset \mathcal{M} \prec \mathbb{U}$ where A is a small base set. Fix a free variable context x, and observe that $L_x(\mathcal{M})$ may be identified with a Boolean subalgebra of $L_x(\mathbb{U})$.

Suppose $I \subset L_x(\mathcal{M})$ is an arbitrary ideal. In general, there is no guarantee that I forms an ideal of the algebra $L_x(\mathbb{U})$: after all, an $\mathcal{L}(\mathcal{M})$-formula may be implied by an $\mathcal{L}(\mathbb{U})$-formula which is not equivalent to any $\mathcal{L}(\mathcal{M})$-formula. One possible remedy is to consider the ideal of $L_x(\mathbb{U})$ generated by I; though, *a priori*, it is unclear whether such an ideal retains any of the definability properties that I bears.

Part of the issue lies in the fact that I is allowed to be an arbitrary ideal in $L_x(M)$; describing such an ideal and using such a description to extend I to $L_x(\mathbb{U})$ may demand more expressive power than the first-order theory of M can afford. If, however, we impose limitations on the descriptive complexity of I, the picture is much different: there is a canonical way to extend I to $L_x(\mathbb{U})$ which recovers I as $I(\mathbb{U}) \cap L_x(M)$—provided M is sufficiently saturated. By "descriptive complexity of I," we mean the descriptive complexity of the sets

$$\{b \in \mathcal{M}_z : \theta(x, b) \in I\}$$

as $\theta(x, z)$ ranges over $\mathcal{L}$-formulas. In the present work we consider three distinct, but related, limitations on the complexity of such sets. Let $\mathcal{M} \prec \mathcal{N}$.

Definition 1.8. I is *definable* over A if for each $\mathcal{L}$-formula $\theta = \theta(x, z)$, the set $\{b \in \mathcal{M}_z : \theta(x, b) \in I\}$ is A-definable. This definable set is denoted $Ix.\theta(x, z)$.

Let $I(\mathcal{N})$ denote the following set of (equivalence classes of) formulas:

$$\Big\{\theta(x, n) : \theta = \theta(x, y) \in \mathcal{L}, n \in \mathcal{N}_y, \mathcal{N} \models Ix.\theta(x, n)\Big\}.$$

Definition 1.9. Suppose $\mathcal{M}$ is $|\mathcal{L}| + |A|^+$-saturated.

(i) I is $\bigwedge$-*definable* over A if for every $\mathcal{L}$-formula $\theta(x, z)$, $\{b \in \mathcal{M}_z : \theta(x, b) \in I\}$ is type-definable over A, i.e. the solution set of some partial type over A. This partial type is denoted $Ix.\theta(x, z)$. Let $I(\mathcal{N})$ denote the following set of (equivalence classes of) formulas:

$$I(\mathcal{N}) = \Big\{\theta(x, n) : \theta = \theta(x, y) \in \mathcal{L}, n \in \mathcal{N}_y, \mathcal{N} \models Ix.\theta(x, n)\Big\}.$$

(ii) I is $\bigvee$-*definable* over A if for every $\theta(x,z)$, $\{b \in \mathcal{M}_z : \theta(x,b) \notin I\}$ is type-definable over A. This partial type is denoted $\neg Ix.\theta(x,z)$. Let $I(\mathcal{N})$ denote the following set of (equivalence classes of)[1] formulas:

$$I(\mathcal{N}) = \left\{\theta(x,n) : \theta = \theta(x,y) \in \mathcal{L}, n \in \mathcal{N}_y, \mathcal{N} \not\models \neg Ix.\theta(x,n)\right\}.$$

Theorem 1.10. *Suppose either*

(1) I is definable over A, or

(2) $\mathcal{M}$ is $|\mathcal{L}| + |A|^+$-saturated, and I is either $\bigwedge$- or $\bigvee$-definable over A.

Then $I(\mathcal{N})$ is an ideal of $L_x(\mathcal{N})$ and $I = I(\mathcal{N}) \cap L_x(\mathcal{M})$.

Proof. We only illustrate the $\bigvee$-definable case, but the others follow similar reasoning.

Fix $\theta = \theta(x,y)$ and $\psi = \psi(x,y')$; by adding dummy variables, we may assume $y = y'$ without loss of generality. To compress notation, write $\pi_\theta(y) := \neg Ix.\theta(x,y)$ and likewise for ψ. Suppose $\mathcal{N} \models \forall x.\theta(x,n) \to \psi(x,n)$ and $\mathcal{N} \models \pi_\theta(n)$. We want to show $\mathcal{N} \models \pi_\psi(n)$. For a contradiction, assume not. Then there is a finite $q \subset \pi_\psi$ such that $\mathcal{N} \models \neg \bigwedge q(n)$. So the following set of formulas is satisfiable in $\mathcal{N}$:

$$\{\forall x.\theta(x,y) \to \psi(x,y)\} \cup \pi_\theta(y) \cup \{\neg \bigwedge q(y)\}.$$

As $\mathcal{M}$ is $\aleph_0 + |A|^+$-saturated, there is $m \in \mathcal{M}_y$ such that $\mathcal{M} \models \forall x.\theta(x,m) \to \psi(x,m)$, $\mathcal{M} \models \pi_\theta(m)$, yet $\mathcal{M} \models \neg \bigwedge q(m)$. Since m satisfies π_θ, it follows that $\theta(x,m) \notin I$. Hence $\psi(x,m) \notin I$ as well, i.e. $\mathcal{M} \models \pi_\psi(m)$. So m satisfies $\pi_\psi(y)$ and $\neg \bigwedge q(y)$ for some finite $q \subset \pi_\psi$, which is absurd.

A similar compactness argument shows that if $\mathcal{N} \models \pi_\theta(n)$ and $\mathcal{N} \models \pi_\psi(n)$, then $\mathcal{N} \models \pi_{\theta \wedge \psi}(n)$.

Finally, it is easy to see $x \neq x \in I$ and to establish $I = I(\mathcal{N}) \cap L_x(\mathcal{M})$. □

[1] As stated, $I(\mathcal{N})$ is a collection of $\mathcal{N}$-formulas. A compactness argument similar to the one in the proof of Theorem 2.10 shows that if $\models \forall x.\varphi(x,b) \leftrightarrow \psi(x,c)$ and $\not\models \neg Ix.\varphi(x,b)$ then $\not\models \neg Ix.\psi(x,c)$. So we do not run into a situation in which some $\psi(x,c) \in I(\mathcal{N})$ (defined as a collection of formulas) is equivalent to some $\varphi(x,b) \notin I(\mathcal{N})$. It is thus safe to view $I(\mathcal{N})$ as a set of equivalence classes of $\mathcal{L}$-formulas over $\mathcal{N}$, i.e. as a *bona fide* subset of $L_x(\mathcal{N})$. This comment also applies to the definable and $\bigwedge$-definable cases.

2. Local wide stability

2.1. **The rank $R^I(-.\varphi,2)$; wide trees.** Fix a partitioned $\mathcal{L}$-formula $\varphi = \varphi(x,y)$.

Definition 2.1. The *wide φ-2-rank* $R^I(-,\varphi,2) =: R(-)$ is defined on formulas $\psi(x) \in L_x(\mathbb{U})$ by induction on $n < \omega$:

(1) $R(\psi(x)) \geq 0 \iff \psi(x) \notin I$;

(2) $R(\psi(x)) \geq n+1 \iff$ there is an instance $\varphi(x,c) \in L_x(\mathbb{U})$ of φ such that

$$R(\psi(x) \wedge \varphi(x,c)^t) \geq n$$

for each $t \in 2$.

Remark 2.2. The following are readily established by induction on n:

(1) For every $\psi, \theta \in L_x(\mathbb{U})$, if $\psi \vdash \theta$ then $R(\psi) \geq n$ implies $R(\theta) \geq n$.

(2) For every $\psi \in L_x(\mathbb{U})$, if $R(\psi) \geq n$ then $R(\psi) \geq m$ for every $m < n$.

In particular, (3.2.ii) implies that $\{n : R(\psi) \geq n\}$ forms an initial segment of ω. So the following definition makes sense:

Definition 2.3. For $\psi \in L_x(\mathbb{U})$, put

(1) $R(\psi) = -\infty$ if $\psi \in I$,

(2) $R(\psi) = +\infty$ if $R(\psi) \geq n$ for every $n < \omega$,

(3) and otherwise

$$R(\psi) = \inf\{n : R(\psi) \not\geq n+1\} = \sup\{n : R(\psi) \geq n\}.$$

Definition 2.4. For a (partial) type $\pi = \pi(x)$ put

$$R(\pi) = \inf\{R\left(\bigwedge q\right) : q \subset \pi \text{ finite}\}.$$

(3.2) also implies that, if $\varphi(x,c)$ witnesses $R(\psi) \geq n+1$, then each $\varphi(x,c)^t$ is wide. That is, wide φ-2-rank (calculated in $\mathbb{U}$) measures the number of times a definable subset of $\mathbb{U}$ can be iteratively split by wide/co-wide instances of φ (over $\mathbb{U}$). It is natural to construct an appropriate binary tree whose consistency controls the wide φ-2-rank.

In order to do so, we need to clarify what we mean by "consistency." If one wishes to impose no bound on the descriptive complexity of the ideals under consideration, one could do the following: expand $\mathbb{U}$ to a structure $\mathbb{U}^I$ in an enriched language $\mathcal{L}^I$ which hardcodes each predicate of the form "$Ix.\theta(x,z)$" and carries

the obvious interpretation in $\mathbb{U}^I$. In this case, the consistency of an $\mathcal{L}^I$-definable tree in $\mathbb{U}^I$ controls the wide φ-2-rank calculated relative to the $\mathcal{L}$-structure $\mathbb{U}$.

The problem is that we would like to dispense with the dependence on $\mathbb{U}$ and work relative to T. That is, the rank we develop should be absolute across all (sufficiently saturated) models of T. *Prima facie*, the expansion to $\mathbb{U}^{\mathcal{I}}$ could introduce pathologies over which the original theory T has no control.

$\bigvee$- and $\bigwedge$-definable ideals avoid this catastrophe. Let $A \subset \mathbb{U}$ be small. Fix a formula $\psi \in L_x(\mathbb{U})$ and an ordinal α.

Definition 2.5. Let $I \subset L_x(\mathbb{U})$ be an ideal, $\bigvee$-definable over A. $\mathrm{Tree}_\psi[I, \varphi, \alpha]$ denotes the following collection of formulas in the free variables $(y_s : s \in^{\alpha >} 2)$:

$$\bigcup \left\{ \neg \mathcal{I}x.\psi(x) \wedge \bigwedge_{l \in K} \varphi(x, y_{\sigma \restriction l})^{\sigma[l]} : \sigma \in {}^{\alpha}2, K \subset \alpha \text{ finite} \right\}.$$

$\mathrm{Tree}[I, \varphi, \alpha]$ refers to the collection of formulas given by $\mathrm{Tree}_{x=x}[I, \varphi, \alpha]$.

$\mathbb{U}$ *has wide φ-trees of height α in ψ* if it realizes $\mathrm{Tree}_\psi[I, \varphi, \alpha]$ (equivalently, by saturation, if $\mathrm{Tree}_\psi[I, \varphi, \alpha]$ is a type).

Definition 2.6. Let $I \subset L_x(\mathbb{U})$ be an ideal, $\bigwedge$-definable over A. $\mathbb{U}$ *has wide φ-trees of height α in ψ* if there is a tree $\mathbf{b} = (b_s : s \in^{\alpha >} 2)$ such that for each finite $K \subset \alpha$ and $\sigma \in^{\alpha} 2$ there is a finite

$$q_{\sigma,K}(\mathbf{y}) \subset Ix.\psi(x) \wedge \bigwedge_{k \in K} \varphi(x, y_{\sigma \restriction k})^{\sigma[k]}$$

such that $\mathbb{U} \models \neg \bigwedge q_{\sigma,K}(\mathbf{b})$.

Proposition 2.7. *Fix an ideal $I \subset L_x(\mathbb{U})$.*

1. *Suppose I is $\bigvee$- or $\bigwedge$-definable. Then for every $k < \omega$ and $\psi \in L_x(\mathbb{U})$, $R(\psi) \geq k$ if, and only if, $\mathbb{U}$ has wide φ-trees of height k.*
2. *Suppose I is $\bigvee$-definable. Then $R(\psi) = \infty$ if, and only if, $\mathrm{Tree}_\psi[I, \varphi, \omega]$ is satisfiable in $\mathbb{U}$.*

Proof. (i) is immediate from the definitions. For (ii), note that if $R(\psi) = \infty$, then by (i) $\mathrm{Tree}_\psi[I, \varphi, k]$ is satisfiable for every $k < \omega$. By compactness, it follows that $\mathrm{Tree}_\psi[I, \varphi, \omega]$ is satisfiable. As $\mathbb{U}$ is saturated (and noting that $\mathrm{Tree}_\psi[I, \varphi, \omega]$ is a type in only countably many free variables) we conclude that $\mathrm{Tree}_\psi[I, \varphi, 2]$ has a realization in $\mathbb{U}$. □

So far, we have suppressed the dependence of $R^I(-, \varphi, 2)$ on the ambient monster model $\mathbb{U}$, but have not yet shown that its value depends only on T and not our choice of $\mathbb{U}$. The following proposition will justify this apparent abuse of notation.

As we discussed in the previous section, it makes sense to think of a $\bigvee$- or $\bigwedge$-definable I not just as an ideal in the algebra $L_x(\mathbb{U})$, but as a compatible family of ideals in the algebras $L_x(\mathcal{M})$ where $A \subset \mathcal{M} \prec \mathbb{U}$. In this spirit let $R_{\mathcal{M}}(-)$ be the wide φ-2-rank relativized to the model $\mathcal{M}$ with respect to the ideal $I(M)$.

Proposition 2.8. *Let $I \subset L_x(\mathbb{U})$ be $\bigvee$- or $\bigwedge$-definable over a small $A \subset \mathcal{M}$, and let $\psi \in L_x(\mathcal{M})$.*

(1) $R_{\mathcal{M}}(\psi) \leq R_{\mathbb{U}}(\psi)$.

(2) If $\mathcal{M}$ is $|\mathcal{L}| + |A|^+$-saturated, then $R_{\mathcal{M}}(\psi) = R_{\mathbb{U}}(\psi)$.

Proof.

(i) By induction on $n < \omega$, one proves that for all $\psi \in L_x(\mathcal{M})$, if $R_{\mathcal{M}}(\psi) \geq n$ then $R_{\mathbb{U}}(\psi) \geq n$. The base case and inductive steps are both immediate from the definitions.

(ii) We treat the $\bigvee$-definable case; the $\bigwedge$-definable one is similar. To do so, we prove that for all $\psi \in L_x(\mathcal{M})$, $R_{\mathbb{U}}(\psi) \geq n$ implies $R_{\mathcal{M}}(\psi) \geq n$. Suppose $R_{\mathbb{U}}(\psi) \geq n$. By Proposition 3.7, this is equivalent to the satisfiability of the partial type $\mathrm{Tree}_\psi[I, \varphi, n]$, defined over A. Since $A \subset \mathcal{M}$ and $\mathcal{M}$ is $|\mathcal{L}| + |A|^+$-saturated, it follows that $\mathrm{Tree}_\psi[I(\mathcal{M}), \varphi, n]$ is realized in $\mathcal{M}$, i.e. $R_{\mathcal{M}}(\psi) \geq n$. Combining this result with (i) delivers the desired conclusion. □

2.2. **The wide unstable formula theorem.** We give several equivalent characterizations of boundedness of wide local 2-rank with respect to a $\bigvee$-definable ideal in several ways—namely, in terms of counting wide local types and definability of wide local types over ω-saturated models.

Fix a partitioned formula $\varphi = \varphi(x, y)$ and let $I \subset L_x(\mathbb{U})$ be a $\bigvee$-definable ideal. To improve presentation, we assume its base is $\emptyset$, although permitting an arbitrary (small) base presents no essential difficulty.

Theorem 2.9. *The following are equivalent:*

a. For every $\lambda \geq 2^{|T|}$ and $A \subset \mathbb{U}$, $|A| \leq \lambda$ implies $|S^I_\varphi(A)| \leq \lambda$.

b. For some $\lambda \geq \aleph_0$, for every $A \subset \mathbb{U}$, $|A| \leq \lambda$ implies $|S^I_\varphi(A)| \leq \lambda$.

c. Tree$[I, \varphi, \omega]$ is not satisfiable.

d. $R^I(x = x, \varphi, 2) < \infty$.

e. For every ω-saturated $\mathcal{M} \prec \mathbb{U}$ and $p \in S^I_\varphi(\mathcal{M})$, p is $\mathcal{M}$-definable.

Theorem 3.9 justifies the following

Definition 2.10. φ is *I-stable* with respect to T if it satisfies the equivalent conditions of Theorem 3.9. Otherwise φ is *I-unstable*

Proof of Theorem 3.9.

(a)⇒(b) Immediate.

(b)⇒(c) Suppose $\mathrm{Tree}[I, \varphi, \omega]$ is satisfiable and fix $\lambda \geq \aleph_0$. Put

$$\mu = \inf\{\kappa : 2^\kappa > \lambda\},$$

so that $\mu \leq \lambda < 2^\mu$. Our assumption ¬(c) delivers a wide φ-tree $(b_s : s \in {}^{\mu>}2)$ of height μ: indeed, since I is $\bigvee$-definable, it is a consequence of the compactness theorem that the satisfiability of $\mathrm{Tree}[I, \varphi, \omega]$ implies the satisfiability of $\mathrm{Tree}[I, \varphi, \alpha]$ for every ordinal α. By definition, the paths $\sigma \in {}^\mu 2$ determine I-wide (partial) types

$$p_\sigma = \{\varphi(x, b_{\sigma\restriction\alpha})^{\sigma[\alpha]} : \alpha < \mu\}$$

which are pairwise (explicitly) contradictory. Taking wide completions, we obtain

$$|B| \leq \mu \leq \lambda < 2^\mu = |S^I_\varphi(B)|$$

where

$$B = \bigcup\{b_s : s \in {}^{\mu>}2\}.$$

(c)⇒(d) Suppose $R^I(x = x, \varphi, 2) \geq n$ for all $n < \omega$—equivalently, $\mathrm{Tree}[I, \varphi, n]$ is satisfiable for every $n < \omega$. As I is $\bigvee$-definable over $\emptyset$, $\mathrm{Tree}[I, \varphi, \omega]$ is equivalent to a collection of $\mathcal{L}$-formulas in countably many free variables. An application of compactness to our assumption ¬(d) thus shows that $\mathrm{Tree}[I, \varphi, \omega]$ is consistent. (And $\aleph_0 + |\mathrm{dom}\ I|^+$-saturation of $\mathbb{U}$ guarantees a realization in $\mathbb{U}$.)

(d)⇒(e) Suppose $R^I(x = x, \varphi, 2) < \infty$. Fix an ω-saturated model $\mathcal{M} \prec \mathbb{U}$ and let $p \in S^I_\varphi(\mathcal{M})$ be a wide φ-type over $\mathcal{M}$. Let $k = R^I(p, \varphi, 2)$. Since p is wide, $k \geq 0$; by our assumption (d), $k < \omega$. By definition of wide φ-2-rank, there exists a finite $p_0 \subset p$ which satisfies

$$R^I(p_0, \varphi, 2) = k.$$

We claim that for every $a \in \mathcal{M}_y$,

$$\varphi(x, a) \in p \iff \mathrm{Tree}_{p_0 \wedge \varphi(x,a)}[I, \varphi, k] \text{ is satisfiable;}$$

$$\neg\varphi(x, a) \in p \iff \mathrm{Tree}_{p_0 \wedge \neg\varphi(x,a)}[I, \varphi, k] \text{ is satisfiable.}$$

Indeed, if $\varphi(x,a) \in p$, then we have

$$\begin{aligned} k = R^I(p_0, \varphi, 2) &\geq R^I(\bigwedge p_0 \wedge \varphi(x,a), \varphi, 2) \\ &\geq R^I(p \cup \{\varphi(x,a)\}, \varphi, 2) \\ &= R^I(p, \varphi, 2) = k. \end{aligned}$$

So $\mathrm{Tree}_{p_0 \wedge \varphi(x,a)}[I, \varphi, k]$ is satisfiable. Conversely, suppose $\varphi(x,a) \notin p$. As p is a complete φ-type over $\mathcal{M}$, we have $\neg\varphi(x,a) \in p$. By the argument above, it follows,

$$R^I(p_0 \wedge \neg\varphi(x,a), \varphi, 2) = k.$$

If $\mathrm{Tree}_{p_0 \wedge \varphi(x,a)}[I, \varphi, k]$ were satisfiable, i.e. $R^I(p_0 \wedge \varphi(x,a), \varphi, 2) \geq k$, then $\varphi(x,a)$ gives witness to

$$R^I(p_0, \varphi, 2) \geq k + 1$$

which contradicts $R^I(p_0, \varphi, 2) = k$. The same reasoning applies to $\neg\varphi(x,a)$, and we have proved our claim.

But in a sufficiently saturated model, projections commute with direct limits of definable sets. Thus the condition

$$\text{“ } \left(\exists y_s : s \in {}^{k>}2\right) . \mathrm{Tree}_{p_0 \wedge \varphi(x,y)}[I, \varphi, k](\mathbf{y})\text{”}$$

is type-definable over $\mathrm{dom}\,(I) \cup \mathrm{dom}\,(p_0) = \mathrm{dom}\,(p_0) =: M_0$. So there exist partial types $\alpha(y)$, $\beta(y)$ over M_0 such that $M_p^+ = \alpha(\mathcal{M})$ and $M_p^- = \beta(\mathcal{M})$, where

$$\begin{aligned} M_p^+ &= \{a \in \mathcal{M}_y : \varphi(x,a) \in p\}, \\ M_p^- &= \{a \in \mathcal{M}_y : \neg\varphi(x,a) \in p\}. \end{aligned}$$

Clearly we have $\alpha(\mathcal{M}) \cap \beta(\mathcal{M}) = \emptyset$. Since $\mathcal{M}$ is ω-saturated and M_0 is finite, it follows that $\alpha \cup \beta$ is inconsistent. The compactness theorem delivers finite $\alpha_0 \subset \alpha$, $\beta_0 \subset \beta$ such that $\alpha_0 \cup \beta_0$ is inconsistent, i.e.

$$\bigwedge \alpha_0 \vdash \neg \bigwedge \beta_0.$$

Thus $d_p x.\varphi(x,y) := \bigwedge \alpha_0$ is an $\mathcal{L}(\mathcal{M})$-formula with the property that for all $a \in \mathcal{M}_y$, if $\varphi(x,a) \in p$ then $\models d_p x.\varphi(x,a)$ and if $\neg\varphi(x,a) \in p$, then $\models \neg d_p x.\varphi(x,a)$. That is, p is definable over $\mathcal{M}$, as desired.

(e)⇒(a) Assume (e) and fix $\lambda \geq 2^{|T|}$. Let $A \subset \mathbb{U}$ and suppose $|A| \leq \lambda$. A extends to an ω-saturated model $\mathcal{M} \prec \mathbb{U}$ of cardinality no larger than

$$|A| + 2^{|T|} \leq \lambda.$$

For each $p \in S^I_\varphi(A)$, choose a wide extension $p' \in S^I_\varphi(\mathcal{M})$. Composing the injections

$$S^I_\varphi(A) \hookrightarrow S^I_\varphi(\mathcal{M}) \hookrightarrow \mathcal{L}(\mathcal{M}),$$
$$p \mapsto p' \mapsto d_p x.\varphi(x, y),$$

obtains the bound $|S^I_\varphi(A)| \leq |\mathcal{L}(\mathcal{M})| \leq \lambda$.

This completes the proof.

□

2.3. **Wide orders.** The wide φ-2-rank generalizes the classical $R(-, \varphi, 2)$ in a natural way. As shown in the previous section, finite wide φ-2-rank with respect to a $\bigvee$-definable ideal I has equivalent characterizations in terms of counting wide φ-types and definability of wide φ-types over ω-saturated models; compare to the classical unstable formula theorem.

Yet other characterizations of local stability abound, and it seems fitting to see whether, and under what modifications, they adapt to the present context. For instance, the classical unstable formula theorem states in part that $R(x = x, \varphi, 2) < \infty$ if, and only if, φ does not have the order property. What is the right modification to the order property so that its failure is equivalent to the condition $R^I(x = x, \varphi, 2) < \infty$? Having characterized wide stability in terms of counting wide φ-types in the previous section, an application of the Erdös–Makkai theorem suggests the following generalization of the order property relative to an ideal:

Definition 2.11. φ has the *wide order property* (with respect to I) if there exist $a_i \in \mathbb{U}_x$, $b_i \in \mathbb{U}_y$ for $i < \omega$ such that for all $i, j < \omega$ we have $\models \varphi(a_i, b_j)^{\text{if } i<j}$ and $\text{tp}_\varphi(a_i/\{b_j : j < \omega\}$ is I-wide.

In order to make this definition amenable to applications of the standard lemma on stretching sequences of indiscernibles, we introduce the following notation when the ambient ideal I is $\bigvee$-definable.

Notation 2.12. Let X be a linear order and $K \subset X$. Consider the following sets of formulas:

$$H(\varphi, X) = \{\varphi(x_i, y_j)^{\text{if } i<j} : i, j \in X\},$$
$$q_{i,K} = \{\varphi(x, y_j)^{\text{if } i<j} : j \in K\},$$
$$H(I, \varphi, X) = \bigcup \left\{ \neg I x_i. \bigwedge q_{i,K}(x_i) : i < \omega, K \subset X \text{ finite} \right\}.$$

Now, we show that wide instability of φ is equivalent to the wide order property. Let $I \subset L_x(\mathbb{U})$ be a $\bigvee$-definable ideal.

Theorem 2.13. *The following are equivalent:*

a. φ is I-stable.

b. φ does not have the wide order property with respect to I.

Proof.

(b)⇒(a) If (a) fails, there exists $B \subset \mathbb{U}$ such that $|S^I_\varphi(B)| > \aleph_0 \geq |B|$. For each $p \in S^I_\varphi(B)$ consider

$$X_p = \{b \in B_y : \varphi(x, b) \in p\},$$

$$\mathcal{X} = \{X_p : p \in S^I_\varphi(B)\}.$$

As any two distinct elements of $S^I_\varphi(B)$ are explicitly contradictory, we have

$$|\mathcal{X}| = |S^I_\varphi(B)| > \aleph_0 \geq |B|.$$

The Erdös–Makkai theorem delivers, for $i < \omega$, $p_i \in S^I_\varphi(B)$ and $b_i \in B_y$ such that either

(i) for all i, j, $\varphi(x, b_j) \in p_i \iff i < j$, or

(ii) for all i, j, $\varphi(x, b_j) \notin p_i \iff i < j$.

Taking realizations $a_i \models p_i$, (i) says that φ has the wide order property and (ii) says that $\neg\varphi$ does. Since I is $\bigvee$-definable, φ has the I-wide order property if, and only if, $\neg\varphi$ does. So we are done.

(a)⇒(b) Assume φ has the I-wide order property. Since I is $\bigvee$-definable, it follows by compactness that $H(I, \varphi, X)$ is satisfiable for every linear order X. Now let $\lambda \geq \aleph_0$ and let $\kappa = \mathrm{ded}(\lambda)$ be its Dedekind number, i.e.

$$\mathrm{ded}(\lambda) = \sup\{|J| : J \text{ is a linear order with a dense subset of cardinality } \lambda\}.$$

Note that $\lambda < \kappa$.

Let X be a linear order of cardinality κ with a dense subset Q of cardinality λ. Let $(b_r : r \in X)$ realize $H(I, \varphi, X)$. For $r \in X \setminus Q$, consider the partial type

$$p_r(x) = \{\varphi(x, b_q)^{\text{if } r<q} : q \in Q\}$$

and let $B = \bigcup\{b_q : q \in Q\}$. Since $(b_r : r \in X)$ realizes $H(I, \varphi, X)$, each $p_r(x)$ has the I-finite meet property. Thus each $p_r(x)$ extends to a complete I-wide φ-type over B. Moreover, $r \neq r'$ implies $p_r \neq p_{r'}$. We conclude

$$|S^I_\varphi(B)| \geq \kappa > \lambda \geq |B|.$$

As $\lambda \geq \aleph_0$ was arbitrary, it follows, φ is I-unstable. □

2.4. **An example.** Let $\mathcal{L}$ be the first-order language of graphs. Consider the expansion $\mathcal{L}^+$ of $\mathcal{L}$ by a new sort $\mathbb{OF}$ which carries the signature of ordered rings, along with, for each $\mathcal{L}$-formula $\theta = \theta(x, z)$, an $l(z)$-ary function symbol f_θ from the home sort $\mathbb{H}$ to $\mathbb{OF}$. In addition, we include a partial unary operation $\log : \mathbb{OF}_{>0} \to \mathbb{OF}$.

If H is a finite $\mathcal{L}$-structure, $(H, \mathbb{R})$ denotes the expansion of H to the $\mathcal{L}^+$-structure which interprets $\mathbb{H}$ as H, $\mathbb{OF}$ as the ordered real field, log as the base-2 logarithm, and for each $\mathcal{L}$-formula $\theta = \theta(x, y)$ the operation f_θ as the map

$$H_y \to \mathbb{R}, b \mapsto |\theta(H, b)|.$$

Let $(G_n : n < \omega)$ be a sequence of finite (simple, undirected) graphs which is increasing, i.e. $|G_n| < |G_{n+1}|$. Take a nonprincipal ultrafilter $\mathcal{U}$ on ω. Then the ultraproduct $\prod_{n\to\mathcal{U}}(G_n, \mathbb{R})$ has the structure of a continuum-sized (simple, undirected) graph $G = \prod_{n\to\mathcal{U}} G_n$ in the home sort and a nonstandard continuum-sized elementary extension $\mathbb{R}^{\mathcal{U}}$ of the ordered real field with the base-2 logarithm. Keeping with the notaation above, $(G, \mathbb{R}^{\mathcal{U}})$ denotes $\prod_{n\to\mathcal{U}}(G_n, \mathbb{R})$. Moreover, we write $|\theta(x, z)| := f_{\theta(x,z)}(z)$.

Fix a variable x, $l(x) = 1$ of the home sort. Consider

$$I = \{\theta(x, b) \in L_x^+(G, \mathbb{R}^{\mathcal{U}}) : (G, \mathbb{R}^{\mathcal{U}}) \models \log|\theta(x, b)| < (1 - \epsilon)\log|G| \text{ for some } \epsilon \in \mathbb{Q}_{>0}.$$

Basic properties of the coarse pseudofinite dimension on G establish that I is an ideal on $L_x^+(G)$ which is $\bigvee$-definable over $\mathbb{Q}$ in the language $\mathcal{L}^+$.

Example 2.14. Let L_n denote the ladder of height n (in which each side is an independent set) and let K_n denote the complete graph on n vertices. Consider

$$G_n = K_{2^n} \sqcup L_n.$$

According to Los's theorem, the edge relation $E(x, y)$ is unstable in $(G, \mathbb{R}^{\mathcal{U}})$. However, $E(x, y)$ is I-stable in $(G, \mathbb{R}^{\mathcal{U}})$. For a contradiction, assume not. As I is $\bigvee$-definable over a countable set of parameters and $(G, \mathbb{R}^{\mathcal{U}})$ is $\aleph_1$-saturated, our assumption and Theorem 3.13 imply the existence of a sequence $(b_i : i < \omega)$ in G such that for each i,

$$E_i(x) = \{E(x, b_j)^{\text{if } i\leq j} : j < \omega\}$$

is I-FMP. For each j choose a representing sequence $(b_j^n : n < \omega) \in \prod_{n<\omega} G_n$ for b_j. For $N > 0$ put $b_{<N}^n = (b_j^n : j < N)$ and

$$H_N(x_0, \ldots, x_{N-1}, y_0, \ldots, y_{N-1}) := \bigwedge_{i,j<N} E(x_i, y_j)^{\text{if } i\leq j}.$$

Let $N \geq 4$.

Claim. $\{n < \omega : b^n_{<N} \subset L_n\} \in \mathcal{U}$.

Proof of claim. Consider the formula

$$\chi(x) := \forall y.E(x,y) \to (\forall z.z \neq x \wedge E(y,z) \to E(x,z)).$$

Notice that for each n, $G_n \models \chi(v)$ if, and only if, $v \in K_{2^n}$. For a contradiction, assume the claim fails, so that

$$N_1 = \{n < \omega : b^n_{<N} \cap K_{2^n} \neq \emptyset\} \in \mathcal{U}.$$

By our choice of $b_{<N}$, an application of Los's theorem gives

$$N_2 = \{n < \omega : G_n \models \exists \mathbf{x}.H_N(\mathbf{x}, b^n_{<N})\} \in \mathcal{U}.$$

Fix $n \in N_1 \cap N_2$, so that $G_n \models \exists \mathbf{x}.H_N(\mathbf{x}, b^n_{<N})$ and for some $j < N$, $G_n \models \chi(b^n_j)$. But any two vertices in an instance of $H_N(\mathbf{x}, \mathbf{y})$ are joined by a path, so in fact for every $j < N$ we must have $G_n \models \chi(b^n_j)$, i.e.

$$N_1 \cap N_2 = \{n < \omega : G_n \models \exists \mathbf{x}.H_N(\mathbf{x}, b^n_{<N}) \text{ and } b^n_{<N} \subset K_{2^n}\}.$$

Moreover if $n \in N_1 \cap N_2$ and $G_n \models H_N(a^n_{<N}, b^n_{<N})$ then the same reasoning shows $a^n_{<N} \subset K_{2^n}$ as well. But as long as $N \geq 2$, $H_N(\mathbf{x}, \mathbf{y})$ will stipulate at least one non-edge requirement between the $a^n_{<N}$ and $b^n_{<N}$; yet $a^n_{<N}$ and $b^n_{<N}$ are contained in K_{2^n}. This is absurd, and the proof is complete. □

Now let n be such that

$$(\dagger) \qquad G_n \models \exists \mathbf{x}.H_N(\mathbf{x}, b^n_{<N}) \text{ and } b^n_{<N} \subset L_n.$$

We want to estimate

$$\frac{\log |H_N(G^N_n, b^n_{<N})}{\log |G^N_n|}.$$

For each $i < N-1$, $|E_i(G_n, b^n)| \leq n$ holds whenever n satisfies (†). For $i = N-1$, $E_{N-1}(G_n, b^n_{<N})| \leq 2^n$. Thus,

$$\begin{aligned}
\frac{\log |H_N(G^N_n, b^n_{<N})|}{\log |G^N_n|} &\leq \frac{\log(n^{N-1}) + \log(2^n)}{\log\left((2^n + 2n)^N\right)} \\
&\leq \frac{N-1}{N} \frac{\log n}{\log(2^n + 2n)} + \frac{1}{N} \\
&\leq \frac{\log n}{\log(2^n + 2n)} + \frac{1}{4}.
\end{aligned}$$

Let M be such that $n \geq M$ implies

$$\frac{\log n}{\log(2^n + 2n)} < \frac{1}{4}.$$

If we put

$$N_3 = \{n < \omega : G_n \models \exists \mathbf{x}. H_N(\mathbf{x}, b^n_{<N}) \text{ and } b^n_{<N} \subset L_n\} \cap [M, \omega)$$
$$N_4 = \left\{ n < \omega : (G_n, \mathbb{R}) \models \frac{\log |H_N(G^N_n, b^n_{<N})}{\log |G^N|} < \frac{1}{2} \right\}$$

then what we just showed amounts to $N_3 \subset N_4$. $N_3 \in \mathcal{U}$ follows from the claim above; hence $N_4 \in \mathcal{U}$ holds as well. Let's see what $(G, \mathbb{R}^{\mathcal{U}})$ has to say about: recalling that $(b_j : j < \omega)$ is a witness to the I-wide order property for the edge relation $E(x, y)$ in the structure $(G, \mathbb{R}^{\mathcal{U}})$, for each $N < \omega$ we have

$$\text{st}\left(\frac{\log |H_N(G^N, b_{<N})|}{\log |G^N|}\right) = \frac{1}{N}\delta\left(\prod_{i<N} E_i(G, b_{<N})\right) = \frac{1}{N}\sum_{i<N} \delta\left(E_i(G, b_{<N})\right) = 1,$$

by definition of $\mathcal{I}$. In particular,

$$(G, \mathbb{R}^{\mathcal{U}}) \models \frac{\log |H_N(G^N, b_{<N})|}{\log |G^N|} \geq \frac{1}{2}$$

holds. So by Los's theorem it follows,

$$\left\{ n < \omega : (G_n, \mathbb{R}^{\mathcal{U}}) \models \frac{\log |H_N(G^N_n, b^n_{<N})|}{\log |G^N|} \geq \frac{1}{2} \right\} \in \mathcal{U}.$$

We thus conclude $\emptyset \in \mathcal{U}$, a contradiction.

3. Global wide stability

3.1. **The product $I \otimes I'$.** Fix free variable contexts x, y and let $I \subset L_x(\mathbb{U})$, $I' \subset L_y(\mathbb{U})$ be ideals, invariant over some small base set A. To ease notation we take $A = \emptyset$.

Definition 3.1. The product $I \otimes I'$ consists of all definable sets $D \subset \mathbb{U}_x \times \mathbb{U}_y$ for which there exists a (definable) $F \in I$ such that

$$\{a \in \mathbb{U}_x : D_a \notin I'\} \subset F.$$

Proposition 3.2.

i. $I \otimes I' \subset L_{xy}(\mathbb{U})$ *is an ideal.*[2]

ii. If I, I' *are* $\bigvee$*-definable over* A*, then so is* $I \otimes I'$*.*

Proof. For (i), note that if F_0 and F_1 witness $D_0, D_1 \in I \otimes I'$, respectively, then $F_0 \cup F_1 \in I'$ witnesses $D_0 \cup D_1 \in I \otimes I'$. Moreover, if F witnesses $D \in I \otimes I'$ and $D' \subset D$, then F still witnesses $D' \in I \otimes I'$. Finally $\emptyset \in I \otimes I'$ is clear.

For (ii), it suffices to prove the following claim: (the class of) $\varphi(x, y, c) \in L_{xy}(\mathbb{U})$ is in $I \otimes I'$ if and only if there exists a formula $\psi(x, z) \in \mathcal{L}$, a finite $p(z) \subset \neg Ix.\psi(x, z)$, and a finite $q(x, w) \subset \neg I'y.\varphi(x, y, w)$ such that

$$\models \exists z.\neg \bigwedge p(z) \wedge \forall x. \bigwedge q(x, c) \to \psi(x, z).$$

For the forward direction, suppose $\varphi(x, y, c) \in I \otimes I'$. Then by definition there exists $\psi(x, c) \in I$ such that for all $a \in \mathbb{U}_x$, if $\varphi(a, y, c) \notin I'$, then $\models \psi(a, d)$. That is,

$$\neg I'y.\varphi(x, y, c) \vdash \psi(x, d).$$

By compactness, there exists a finite $q(x, w) \subset \neg I'y.\varphi(x, y, w)$ such that

$$q(x, c) \vdash \psi(x, d).$$

Moreover, as $\psi(x, d) \in I$, there is $p(z) \subset \neg Ix.\psi(x, z)$ such that $\models \neg \bigwedge p(d)$. Thus there exist $\psi(x, z)$, $p(z)$, and $q(x, w)$ such that

$$\models \neg \bigwedge p(d) \wedge \forall x. \bigwedge q(x, c) \to \psi(x, d),$$

as desired. The other direction is trivial.

□

[2] In [2] Hrushovski defines $I \otimes I'$ as the ideal generated by definable sets $D \subset L_{xy}(\mathbb{U})$ for which the set of $a \in \mathbb{U}$ with I'-wide fibers D_a has an I-thin (definable cover). However, it appears this set already forms an ideal of $L_{xy}(\mathbb{U})$.

Proposition 3.3. *(cf. [2], pp. 9-10)*

i. If a/A is I-wide and b/Aa is I'-wide, then ab/A is $I \otimes I'$-wide.

ii. If ab/A is $I \otimes I'$-wide then a/A is I-wide. If, in addition, I' is $\bigvee$-definable (over $\emptyset$), then b/Aa is I'-wide.

Proof. First, suppose ab/A is $I \otimes I'$-thin; then there is an A-definable set $X \subset \mathbb{U}_{xy}$ and a definable $F \in I$ such that $ab \in X$ and $\{c \in \mathbb{U}_x : X_c \notin I'\} \subset F$. We want to show a/A is I-thin or b/Aa is I'-thin; so suppose b/Aa is I'-wide. Then $b \in X_a \notin I'$, which implies $a \in F$ by definition. Since I' is invariant (over $\emptyset$), it follows that $\mathrm{tp}(a/A) \vdash x \in F$. Compactness thus delivers an A-definable set $G \ni a$ implying F. In particular, $G \in I$ and $G \in \mathrm{tp}(a/A)$, so a/A is I-thin.

For the converse, first suppose a/A is I-thin. Then $a \in X$ for some I-thin, A-definable set X. Putting $Y = \{cd \in \mathbb{U}_{xy} : c \in X\}$ obtains

$$\{c \in \mathbb{U}_x : Y_x \notin I'\} \subset X \in I.$$

Since Y is A-definable and $ab \in Y$, it follows that ab/A is $I \otimes I'$-thin.

Now suppose I' is $\bigvee$-definable (over $\emptyset$) and assume that b/Aa' is I'-thin. Let $\varphi(x, y) \in \mathcal{L}(A)$ be such that $\models \varphi(a, b)$ and $\varphi(a, \mathbb{U}) \in I'$. Because I' is $\bigvee$-definable, there is a formula $\psi(x) \in \mathcal{L}$ such that $\varphi(c, \mathbb{U}) \in I'$ if, and only if, $\models \psi(c)$. Consider the set $Y \subset \mathbb{U}_{xy}$ defined by $\varphi(x, y) \wedge \psi(x)$. Then Y is A-definable and $ab \in Y$. Moreover, we have

$$\{c \in \mathbb{U}_x : Y_c \notin I'\} = \emptyset$$

because for any c, if $\models \psi(c)$, then $Y_c = \varphi(c, \mathbb{U}) \in I'$ by our choice of $\psi(x)$, and if $\models \neg\psi(c)$ then $Y_c = \emptyset \in I'$. So $Y \in I \otimes I'$, and we conclude that ab/A is $I \otimes I'$-thin, as desired. □

Definition 3.4. The ideals $I^{\otimes n}$ are defined by induction on $n > 0$:

$$I^1 = I,$$

$$I^{\otimes(n+1)} = I^{\otimes n} \otimes I.$$

Corollary 3.5. *Let $l(x) = 1$ and suppose I is $\bigvee$-definable (over $\emptyset$). Let $a \in \mathbb{U}^n$, $b \in \mathbb{U}$. Then ab/A is $I^{\otimes n} \otimes I$-wide if, and only if, a/A is $I^{\otimes n}$-wide and b/Aa is I-wide.*

3.2. **Counting wide $\mathcal{L}$-types.** Fix a free variable context x, $l(x) = 1$ and let $I \subset L_x(\mathbb{U})$ be an ideal, $\bigvee$-definable over $\emptyset$.

Theorem 3.6. *The following are equivalent:*

a. Every $\varphi = \varphi(x, y)$ is $I^{\otimes l(x)}$-stable.

b. Every $\varphi = \varphi(x, y)$ with $l(x) = 1$ is I-stable.

c. For every $\lambda = \lambda^{|T|}$, $|A| \leq \lambda$ *implies* $|S_1^I(A)| \leq \lambda$.

d. For every $n > 0$, $\lambda = \lambda^{|T|}$, $|A| \leq \lambda$ *implies* $|S_n^{I^{\otimes n}}(A)| \leq \lambda$.

Definition 3.7. T is *I-stable* if the equivalent conditions of Theorem 4.7 hold.

The proof of Theorem 4.7 relies on the following

Lemma 3.8. *Consider the restriction* $res : S_{n+1}(A) \to S_n(A)$, $ab/A \mapsto a/A$.

i. res maps $S_{n+1}^{I^{\otimes n+1}}(A)$ *onto* $S_n^{I^{\otimes n}}(A)$.

ii. If a/A *is* $I^{\otimes n}$*-wide, then we have a bijection*

$$S_1^I(Aa) \to res^{-1}(a/A) \cap S_{n+1}^{I^{\otimes n+1}}(A),$$

$$b/Aa \mapsto ab/A$$

Proof. If ab/A is $I^{\otimes n+1}$-wide, then a/A is $I^{\otimes n}$-wide by Proposition 4.3. On the other hand, if a/A is $I^{\otimes n}$-wide, take b so that b/Aa is I-wide; then by the same proposition, ab/A is $I^{\otimes n+1}$-wide. This proves the first item.

For the second, note that Proposition 4.3 guarantees the map in question is well-defined. It is clearly injective. For surjectivity, suppose $a'b'/A \in \text{res}^{-1}(a/A) \cap S_{n+1}^{I^{\otimes n+1}}(A)$. Then, in particular, we have $a' \equiv_A a$, so there is an automorphism $\sigma \in \text{Aut}(\mathbb{U}/A)$ mapping a' to a. Putting $b = \sigma(b')$ obtains $a'b'/A = ab/A$. Now, since $a'b'/A$ is $I^{\otimes n+1}$-wide, Proposition 4.3 guarantees b'/Aa' is I-wide. It follows that $b/Aa = \sigma(b'/Aa')$ is I-wide, as I is invariant (over $\emptyset$). This completes the proof. □

Proof of Theorem 4.7.

(a)⇒(b) Trivial.

(b)⇒(c) Fix $\lambda = \lambda^{|T|}$ and consider the map

$$S_1^I(A) \to \prod_{\varphi = \varphi(x,y)} S_\varphi^I(A)$$

$$p \mapsto (p \restriction \varphi : \varphi = \varphi(x,y) \in \mathcal{L}).$$

This map is well-defined because if $p \in S_1(A)$ is I-wide, then so are all of its restrictions to φ-types. Moreover, it is clearly injective. We assumed every $\varphi = \varphi(x,y)$, $l(x) = 1$, is I-stable. Seeing that $\lambda = \lambda^{|T|}$ implies $\lambda \geq 2^{|T|}$, it follows by Theorem 3.9 that if $|A| \leq \lambda$, then $|S_\varphi^I(A)| \leq \lambda$. Therefore we conclude

$$\left|S_1^I(A)\right| \leq \lambda^{|T|} = \lambda,$$

as desired.

(c)⇒(d) The proof is by induction on $n > 0$. The base case $n = 1$ is our assumption (c). Assume the result holds for all $0 < k \leq n$. Let $\lambda = \lambda^{|T|}$ and fix A with $|A| \leq \lambda$. We want to show

$$\left|S_{n+1}^{I^{\otimes n+1}}(A)\right| \leq \lambda.$$

Keeping in mind Lemma 4.9.i, we may view the restriction map $\text{res} : ab/A \mapsto a/A$ as a surjection

$$S_{n+1}^{I^{\otimes n+1}} \to S_n^{I^{\otimes n}}(A)$$

and write

$$S_{n+1}^{I^{\otimes n+1}}(A) = \bigcup\{\text{res}^{-1}(p) : p \in S_n^{I^{\otimes n}}(A)\}.$$

Applying the inductive hypothesis $k = n$ to λ and A gives $|S_n^{I^{\otimes n}}(A)| \leq \lambda$. So it suffices to show for each $p \in S_n^{I^{\otimes n}}(A)$, $|\text{res}^{-1}(p)| \leq \lambda$. Fix such a p and let a be a realization. By Lemma 4.9.ii, it follows,

$$|\text{res}^{-1}(p)| = \left|S_1^I(Aa)\right|.$$

The inductive hypothesis $k = 1$ applied to λ and Aa implies $\left|S_1^I(Aa)\right| \leq \lambda$, which is what we want to show.

(d)⇒(a) Let $\varphi = \varphi(x, y)$ be a partitioned $\mathcal{L}$-formula and let $l = l(x)$. In view of Theorem 3.9 and recalling that each power of I is $\bigvee$-definable over $\emptyset$, it suffices to show there exists $\lambda \geq 2^{|T|}$ such that $\left|S_\varphi^{I^{\otimes l}}(A)\right| \leq \lambda$ whenever $|A| \leq \lambda$. Let $\lambda = \lambda^{|T|}$. (In particular, we have $\lambda \geq 2^{|T|}$.) Fix A with $|A| \leq \lambda$. For each $p \in S_\varphi^{I^{\otimes l}}(A)$, choose a wide completion $\tilde{p} \in S_l^{I^{\otimes l}}(A)$. The assignment $p \mapsto \tilde{p}$ is an injection $S_\varphi^{I^{\otimes l}}(A) \to S_l^{I^{\otimes l}}(A)$. So (d) implies, in particular, $\left|S_\varphi^I(A)\right| \leq \lambda$, as desired. □

3.3. ***I*-dividing.** Fix an ($\emptyset$-)invariant ideal $I \subset L_x(\mathbb{U})$. Let $\varphi(x, c) \in L_x(\mathbb{U})$ and let $A \subset \mathbb{U}$ be small.

Definition 3.9. $\varphi(x, c)$ *I-divides* over A if there is an A-indiscernible sequence $(c_i : i < \omega)$ in c/A such that $\{\varphi(x, c_i) : i < \omega\}$ is I-thin, i.e. there exists a finite $K \subset \omega$ such that $\bigwedge_{i \in K} \varphi(x, c_i) \in I$.

A partial type $\pi(x)$ *I-divides* over A if it implies a formula which I-divides over A.

Observation 3.10.

i. Since $(c_i)_{i<\omega}$ is A-indiscernible, $\{\varphi(x, c_i)\}$ is I-thin if and only if there exists $k < \omega$ such that $\bigwedge_{i \in K} \varphi(x, c_i) \in I$ holds for all $K \in \binom{\omega}{k}$.

ii. $\pi(x)$ I-divides over A if and only if there is a finite $\pi_0(x) \subset \pi(x)$ such that $\bigwedge \pi_0(x)$ I-divides over A.

iii. If $p \in S^I(\mathbb{U})$ is A-invariant, then p does not I-divide over A. In particular, the conclusion holds if p is A-definable.

Proposition 3.11. *If I is $\bigwedge$-definable, then $\varphi(x, c)$ I-divides over A if and only if there exist $k < \omega$ and a sequence $(c_i : i < \omega)$ in c/A such that $\bigwedge_{i \in K} \varphi(x, c_i) \in I$ for all $K \in \binom{\omega}{k}$.*

Proof. The forward implication is immediate. For the converse, let $\mathbf{c} = (c_i : i < \omega)$ be a sequence in c/A and $k < \omega$ such that $\bigwedge_{i \in K} \varphi(x, c_i) \in I$. Since I is $\bigwedge$-definable, it follows that $Ix. \bigwedge_{i<k} \varphi(x, y_i)$ is contained in the Ehrenfeucht–Mostowski type $\mathcal{EM}(\mathbf{c}/A)$. Applying the standard lemma to $\mathbf{c}$ thus delivers an A-indiscernible $(c'_i : i < \omega)$ such that $\{\varphi(x, c'_i) : i < \omega\}$ is I-thin. □

Proposition 3.12. *If I is $\bigvee$-definable, then the following are equivalent:*

a. *a/Ab does not I-divide over A.*
b. *For every A-indiscernible sequence $\mathbf{b} \ni b$ there exist $a' \equiv_{Ab} a$ and $\mathbf{b}' \equiv_{Ab} \mathbf{b}$ such that $\mathbf{b}'$ is Aa'-indiscernible and $a'/A\mathbf{b}'$ is I-wide.*
c. *For every A-indiscernible sequence $\mathbf{b} \ni b$ there exists $\mathbf{b}' \equiv_{Ab} \mathbf{b}$ such that $\mathbf{b}'$ is Aa-indiscernible and $a/A\mathbf{b}'$ is I-wide.*
d. *For every A-indiscernible sequence $\mathbf{b} \ni b$ there exists $a' \equiv_{Ab} a$such that $\mathbf{b}$ is Aa'-indiscernible and $a'/A\mathbf{b}$ is I-wide.*

Proof. (c)⇒(b) and (d)⇒(b) are trivial. For (b)⇒(c), let $\mathbf{b} \ni b$ be A-indiscernible. Take a', $\mathbf{b}'$ as given by (b). Then there is $\sigma \in \mathrm{Aut}(\mathbb{U}/Ab)$ which sends $\mathbf{b}'$ to $\mathbf{b}$. Put $a'' = \sigma(a')$. Since $\mathbf{b}'$ is Ab-indiscernible, $\mathbf{b} = \sigma(\mathbf{b}')$ is indiscernible over $\sigma(Aa') = Aa''$. Clearly $a'' \equiv_{Ab} a$. Moreover, as I is invariant and a'/Ab', so is $a''/Ab = \sigma \cdot a'/Ab$. The argument for (d)⇒(b) is similar.

Write $p(x, y) = \mathrm{tp}(ab/A)$.

(c)⇒(a): Let $\mathbf{b} \ni b$ be A-indiscernible. (c) delivers $a' \equiv_{Ab} a$ such that $\mathbf{b}$ is Aa'-indiscernible and $a'/A\mathbf{b}$ is wide. Then, as $\models p(a, b)$ and $\mathbf{b}'$ is Aa'-indiscernible, $\bigcup_i p(x, b_i)$ is contained in $\mathrm{tp}(a'/A\mathbf{b})$, which is wide. So $\bigcup_i p(x, b_i)$ is also wide, as desired.

For (a)⇒(b), we will require a modified version of the standard lemma, stated and proved below. Let $\mathbf{b} = (b_i : i \in J) \ni b =: b_{i_0}$ be A-indiscernible. Then, by assumption, $\bigcup_{i \in J} p(x, b_i)$ is I-wide; so there is a realization a' so that $a'/A\mathbf{b}$ is wide. The wide standard lemma delivers an Aa'-indiscernible sequence $\mathbf{b}'' = (b''_i : i \in J)$ which realizes $\mathcal{EM}(\mathbf{b}/Aa')$ and is such that $a'/A\mathbf{b}''$ is wide. In particular, we have $\models p(a', b_{i_0}) \cup p(a', b''_{i_0})$, so there is $\sigma \in \mathrm{Aut}(\mathbb{U}/Aa')$ sending b''_{i_0} to b_{i_0}.

Let $\mathbf{b}' = \sigma(\mathbf{b}'')$. Since $\mathbf{b} \ni b$, $\models p(a', b)$, i.e. $a' \equiv_{Ab} a$, holds. Since $\mathbf{b}''$ is Aa'-indiscernible, so is $\mathbf{b}' = \sigma(\mathbf{b}'')$. Since $a'/A\mathbf{b}''$ is wide, so is $a'/A\mathbf{b}' = \sigma \cdot a'/A\mathbf{b}''$. Finally, since

$$\mathbf{b}'' = \sigma^{-1}(\mathbf{b}') \models \mathcal{EM}(\mathbf{b}/Aa') \supset \mathcal{EM}(/A),$$

$b \in \mathbf{b}$, $b \in \mathbf{b}'$, and $\mathbf{b}$, $\mathbf{b}'$ are A-indiscernible, we have $\mathbf{b}' \equiv_{Ab} \mathbf{b}$. This completes the proof. □

Lemma 3.13. *(Wide Standard Lemma) Let $I \subset L_x(\mathbb{U})$ be a $\bigvee$-definable ideal. Let J, K be infinite linear orders, $\mathbf{b} \in {}^J\mathbb{U}_y$, and $a \in \mathbb{U}_x$. Suppose $a/A\mathbf{b}$ is I-wide. Then there is an Aa-indiscernible sequence $\mathbf{b}' \in {}^K\mathbb{U}_y$ which realizes $\mathcal{EM}(\mathbf{b}/Aa)$ and is such that $a/A\mathbf{b}'$ is I-wide.*

Proof. We want to realize $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \{\varphi(y_{k_1}, \dots, y_{k_n}) \leftrightarrow \varphi(y_{k'_1}, \dots, y_{k'_n}) : n < \omega, k_1^{(\prime)} < \dots < k_n^{(\prime)} \in K, \varphi(\mathbf{y}) \in \mathcal{L}(Aa)\},$$

$$\Gamma_2 = \{\varphi(y_1, \dots, y_k) : n < \omega, k_1 < \dots < k_n \in K, \varphi(\overline{y}) \in \mathcal{EM}(\mathbf{b}/Aa)\},$$

$$\Gamma_3 = \{\neg\theta(a, \mathbf{y}) \vee \bigwedge \pi(\mathbf{y}) : \theta(x, \mathbf{y}) \in \mathcal{L}(A), \pi(\mathbf{y}) \subset \neg Ix.\theta(x, \mathbf{y}) \text{ finite}\}.$$

Let $\Delta \subset \Gamma$ be finite and write

$$\Delta_1 = \Delta \cap \Gamma_1 = \{\varphi_{1,i}(\mathbf{y}_{1,i}) \leftrightarrow \varphi_{1,i}(\mathbf{y}'_{1,i}) : i < n_1\},$$

$$\Delta_2 = \Delta \cap \Gamma_2 = \{\varphi_{2,i}(\mathbf{y}_{2,i}) : i < n_2\},$$

$$\Delta_3 = \Delta \cap \Gamma_3 = \{\neg\theta_{3,i}(a, \mathbf{y}_{3,i}) \vee \bigwedge \pi_{3,i}(\mathbf{y}_{3,i}) : i < n_3\}.$$

By padding the formulas with dummy variables we may assume that each of the formulas written above is in the same free variables $\mathbf{y}$. Let $l = l(\mathbf{y})$. Consider the ${}^{n_1}2$-coloring of the increasing l-length subsequences of $\mathbf{b}$ given by

$$\mathbf{b}_0 \mapsto \{i < n :\models \varphi_{1,i}(\mathbf{b}_0)\}.$$

By Ramsey's theorem, there is an infinite subsequence $\mathbf{b}_*$ of $\mathbf{b}$ so that the set of increasing l-length subsequences of $\mathbf{b}_*$ is monochromatic. Thus $\mathbf{b}_*$ satisfies Δ_1. Since $\mathbf{b}_*$ is a subsequence of $\mathbf{b}$, it satisfies Δ_2; moreover, since $a/A\mathbf{b}$ is I-wide, it satisfies Δ_3. This completes the proof. □

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