Math DRP Spring 25 Summary - Algebraic Geometry

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During this spring semester DRP, we worked through the paper, "On Algebraic Geometry and the 27 Lines on Smooth Cubic Surfaces in $\mathbb{C}P^3$" written by Basten Dekoninck and Thibeau Wouters. We made it through the majority of the paper, ending with the proof that any cubic surface in $\mathbb{C}P^3$ contains at least one straight line. This is my attempt to briefly summarize everything we went over this semester.

We began by covering some of the required basic background knowledge of Algebra and Algebraic Geometry. We introduced the correspondence V that takes an ideal in the ring $K[x_1, \dots, x_n]$ to the set of all points in $A^n(K)$ that the ideal vanishes on. We also introduced the correspondence I that takes a subset of $A^n(K)$ to the ideal consisting of polynomials in $K[x_1, \dots, x_n]$ that vanish on the set. We also introduced the general definition of a topology (through closed sets), and proved that that the collection of all Algebraic subsets of $A^n(K)$ (ie, subsets that can be written as $V(I)$ with I an ideal) forms a topology (namely the Zariski topology). We also defined a Noetherian ring, demonstrated that $K[x_1, \dots, x_n]$ is Noetherian, and also demostrated that for a Noetherian ring R, R/I is also Noetherian. Importantly, a consequence of this is that every algebraic subset corresponds to an ideal generated by a finite number of polynomials. Finally, we demonstrated that there is a one-to-one correspondence between prime ideals and irreducible algebraic subsets of $A^n(K)$, and made the definition that a set is called an *affine variety*.

Next, we began translating these ideas from $A^{n+1}(K)$ to KP^n. We first made the definition of a projective space, defining an equivalence relation on $A^{n+1}(K) \setminus \{0\}$ by

$$p \equiv q \iff \text{there exists a nonzero } \lambda \in K \text{ s.t. } p = \lambda q$$

and then defined KP^n as $(A^{n+1}(K) \setminus \{0\})/\equiv$. We then defined a homogeneous polynomial in $K[x_0, \dots, x_n]$ of degree d as a polynomial s.t. every non-zero monomial has degree d. For such a polynomial f, it is then the case that $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$. After, this definition, we then defined a homogeneous ideal J as an ideal in which every element in J can be written as a sum of homogeneous polynomials in J (equivalently, the ideal is generated by homogeneous polynomials). We can then define a projective version of the correspondence V which takes a homogeneous ideal in $K[x_0, \dots, x_n]$ to the set of all points in KP^n that vanish on the ideal (the fact that this is a homogeneous ideal ensures this is well-defined on the equivalence relation we used to construct KP^n). We can also define the corresponding case of I that takes a subset of KP^n to the homogeneous ideal consisting of all polynomials that vanish on the subset (we showed that this is always equal to a homogeneous ideal for any subset of KP^n). All of the topological properties of V from the affine case remain true, hence we can equip KP^n with a topology in the same way we did for $A^{n+1}(K)$. In this case, it is again called the Zariski topology. We can similarly define a *projective variety* as an irreducible algebraic subset of KP^n (in this case, corresponding to a homogeneous prime ideal). There is one difference we need to keep in mind between $A^{n+1}(K)$ and KP^n, namely that consisting of the homogeneous ideal $J = (x_0, \dots, x_n)$. In $A^{n+1}(K)$, $V(J) = \{(0, \dots, 0)\}$, however in KP^n we have $V(J) = \varnothing$ (this is called the *irrelevant ideal* in this case).

Now that we made the important basic definitions, our next goal was to come up with some sort of definition for the dimension of a projective variety. We first considered two motivating examples for this, both in the case of $A^2(\mathbb{R})$. The first one is $V(y - x^2)$, which corresponds to the parabola $y = x^2$ in $\mathbb{R}^2$. In this case, at any point, the vector space consisting of all vectors tangent to the curve at that point is always one-dimensional. Hence, the "dimension" of the variety $V(y - x^2)$ should be one, in our definition. The next example was $V(y^2 - x^3)$ in $A^2(\mathbb{R})$. In this case, the space of tangent vectors to a point that is not $(0,0)$ has dimension 1. However, there is a cusp at $(0,0)$, and in some sense, we can think of this as meaning that any vector in $\mathbb{R}^2$ is tangent to the curve at $(0,0)$, hence the dimension of the tangent space here is 2. However, the dimension of the tangent space of the curve is still 1 "almost everywhere," and so we want our definition to agree that the dimension of this variety is also 1. We formalize this (although in the projective not affine case) below

Definition. Let $V \subseteq KP^n$ be a projective variety, and let $P = [(p_0, \ldots, p_n)]$ in V. For any homogeneous ideal $f \in K[x_0, \ldots, x_n]$, we define

$$f_P^{(1)} = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(P) x_i$$

Note that this is a degree one homogeneous polynomial. We can then define the tangent space to V at P as

$$T_P V = \bigcap_{f \in I(V)} V(f_P^{(1)})$$

Since all ideals are finitely generated, we can assume $I(V) = (f_1, \ldots, f_k)$ and apply the product rule for derivatives and simplify to eventually see that this simply becomes

$$T_P(V) = \bigcap_{i=1}^{k} V(f_{i,P}^{(1)})$$

In the case that the ideal is principal, this simplifies to

$$T_P(V) = V(f_P^{(1)})$$

Now that we have defined the tangent space, we can make another definition.

Definition. $P \in V(f) \subseteq KP^n$ is singular if $\frac{\partial f}{\partial x_i}(P) = 0 \ \forall i$.

More generally, we have that $P \in V(f_1, \ldots, f_k) \subseteq KP^n$ is singular if

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_k}{\partial x_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix} (P)$$

doesn't have full rank (Jacobian criteria).

Proposition. *For a projective variety $V \subseteq KP^n$, there is a unique $r \geq 0$ and an open subset $V_0 \subseteq V$ s.t. $\forall P \in V_0$, $\dim T_P V = r$, and $\forall P \in V$, $\dim T_P V \geq r$.*

Definition. $\dim V = r$ where r is as above.

We next defined morphisms of projective varieties. First, we define rational functions on a projective variety as a partially defined function $f : V \to K$ (with $V \subseteq KP^n$) s.t. there exists homogeneous polynomials $g, h \in K[x_0, \ldots, x_n]$ with $f(P) = g(P)/h(P)$ for any $P \in V$ with $h(P) \neq 0$. This is well defined on the projective space since the polynomials are homogeneous, however, the choice of the two polynomials to represent the rational function is not necessarily unique, leading us to define another equivalence relation on $K[x_0, \ldots, x_n]$ by

$$g/h \equiv g'/h' \iff h'g - g'h \in I(V)$$

which is the same as saying that they define the same rational function. We then define the function field of a projecive variety $V \subseteq KP^n$ by

$$K(V) = \left\{ \frac{g}{h} : g, h \in K[x_0 \ldots, x_n] \text{ homog. with } \deg g = \deg h,\ h \notin I(V) \right\} / \equiv$$

We also say that a rational map is regular at P is there is some choice of representation for which $f = g/h, h(P) \neq 0$. A rational function is regular if it is regular on V.

We can next extend this definition so that we have partially defined functions from V to KP^m instead. We defined a rational map to be a partially defined function $f : V \to KP^m$ with $V \subseteq KP^n$ s.t. we have rational functions $f_0, \dots, f_m \in K(V)$ with $f(P) = [(f_0(P), \dots, f_m(P))]$ for all $P \in V$. We defined regularity similarly. We can finally define a morphism between projective varieties:

Definition. Let $V \subseteq KP^n, W \subseteq KP^m$ be projective varieties. A morphism between V and W is a regular rational map $f : V \to W$

We need to make one more definition.

Definition. For a rational map $f : X \to Y$, it is birational if it has a rational inverse. If f is regular, we then say that f is an isomorphism, and V and W are isomorphic projective varieties.

Below are some propositions about morphisms that will be useful.

Proposition. *If $f : X \to Y$ is a morphism of 2 proj varieties, and X irreducible, then $f(X)$ irreducible.*

Proposition. *If $f : X \to Y$ is an isomorphism of projective varieties, then $\dim X = \dim Y$.*

Proposition 1. *Let $f : X \to Y$ be a morphism of projective varieties with $\dim X = \dim Y$. If $\exists y \in Y$ s.t. $f^{-1}(y)$ is finite, then f is surjective.*

Next, we looked at the Fermat cubic surface, which is the set $S \subseteq \mathbb{C}P^3$ defined by $S = V(x_0^3 + x_1^3 + x_2^3 + x_3^3)$. We then explicitly counted that this surface does indeed have 27 straight lines (in summary, we parametrized straight lines up to permutation of coordinates, showed that this gives us 9 lines, then multiplied by 3 to account for the possible permutation of the coordinates).

The final idea we introduced is the Grassmanian of a vector space.

Definition. Let V be an n-dimensional vector space, and let $0 \leq k \leq n$. Then, the Grassmanian $\mathrm{Gr}(k, V)$ is the collection of all k-dimensional subspaces of V. Since all vector spaces of the same dimension are isomorphic, we instead will denote this as $\mathrm{Gr}(k, n)$.

We also showed that the Grassmanian is a projective variety, and that is has dimension $k(n - k)$. Now, we finally got to demonstrate the existence of a line in a cubic surface.

Proposition. *Let $S \subseteq \mathbb{C}P^3$ be a cubic surface. Then, there is a line $\ell \subseteq S$*

Proof. First, notice that we have a 1-to-1 correspondence between lines in $\mathbb{C}P^3$ and planes through the origin in $A^4(\mathbb{C})$. In addition, planes through the origin in $A^4(\mathbb{C})$ can be thought of as 2-D subspaces of $\mathbb{C}^4$ (as a VS), thus we get a 1-to-1 correspondence between lines in $\mathbb{C}P^3$ and $\mathrm{Gr}(2, 4)$. Next, we considered the parameter space of ALL cubic surfaces in $\mathbb{C}P^3$, and used a counting argument to show that this has dimension $\binom{6}{3} - 1 = 19$. Hence, we can identify this space with $\mathbb{C}P^{19}$. Then, we let Z be the projective variety

$$Z = \{(\ell, S) : \ell \in \mathrm{Gr}(2, 4), S \in \mathbb{C}P^{19}, \ell \subseteq S\}$$

We need to show that $\dim Z = 19$ if our goal is to show 27 lines, which we did next. Let $\pi_1 : Z \to \mathrm{Gr}(2, 4)$ be the morphism of proj varieties that projects onto the first coordinate. Then, for $\ell \in \mathrm{Gr}(2, 4)$, we have that $\dim Z = \dim \mathrm{Gr}(2, 4) + \dim \pi_1^{-1}(\ell)$. Since $\dim \mathrm{Gr}(2, 4) = 4$, we need to show that $\dim \pi_1^{-1}(\ell) = 15$. In summary, if we parametrize a general ℓ inside of a cubic surface S, we lose four degrees of of freedom from 19, giving us dimension 15 as desired. Thus, $\dim Z = 19$. Finally, we show the existance of a line. Consider the morphism $\pi_2 : Z \to \mathbb{C}P^{19}$ that projects onto the second coordinate. We know that $\dim Z = \dim \mathbb{C}P^{19} = 19$. Furthermore, we already showed that the Fermat cubic surface S has 27 lines. Importantly, 27 is finite. Hence, we have some $S \in \mathbb{C}P^{19}$ with $\pi_2^{-1}(S)$ finite. Thus, by Proposition 1, π_2 is surjective, and hence given any cubic S, there is a line ℓ with $\ell \subseteq S$. □