WAVE DECAY: DISPERSION IN THE STATIONARY PHASE

LANEY SAYE

Contents

1. Introduction

The study of waves is a central problem in physics, emerging in fields such as electromagnetism, optics, and quantum mechanics. Many wave phenomena can be modeled by the linear wave equation. In one spatial dimension, this equation takes the form:

$$\left(-\partial_t^2 + \partial_x^2\right) u = 0,$$

where $u(x,t)$ describes the wave's displacement at position x and time t.

Understanding the behavior of solutions to the wave equation is critical for interpreting physical phenomena. In Section 3, we will solve this partial differential equation (PDE) and obtain the general solution:

$$u(x,t) = \int_{\mathbb{R}} \widehat{u_+}(\xi) e^{i(t\xi + x\xi)} \frac{\mathrm{d}\xi}{2\pi} + \int_{\mathbb{R}} \widehat{u_-}(\xi) e^{i(-t\xi + x\xi)} \frac{\mathrm{d}\xi}{2\pi}.$$

A key aspect of wave propagation is its asymptotic behavior, for example, the value of $u(x,t)$ as $t \to \infty$. Over time, waves exhibit dispersion: different frequency components travel at different velocities, causing the wave to spread over a larger spatial region. This redistribution of energy decreases the wave's amplitude.

Building on solutions to the wave equation, we will analyze oscillatory integrals that describe how waves evolve over time. These integrals take the form:

$$I(\lambda) = \int_{\mathbb{R}} a(x) e^{i\lambda\varphi(x)} dx.$$

In Section 4, we aim to study the asymptotic behavior of these integrals in the limit where the parameter $\lambda \to \infty$, motivated by the concept of dispersion.

The stationary phase theorem is a useful tool to approximate these oscillatory integrals, indicating the significance of critical points in the phase function $\varphi(x)$. The stationary phase approximation is given by:

$$I(\lambda) = e^{i\lambda\varphi(x_0)} \left(\frac{2\pi}{|\varphi''(x_0)|}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\varphi''(x_0)}\lambda^{-\frac{1}{2}} \left(a_0 + a_1\lambda^{-1} + \cdots + a_{N-1}\lambda^{-N+1}\right) + \lambda^{-N-\frac{1}{2}}S_N.$$

In investigating this theorem, we will show asymptotic decay of solutions to the wave equation.

1.1. **Outline.** In this paper, we first introduce the Fourier transform and its utility in solving PDEs. We then apply this technique to solve the one-dimensional linear wave equation, focusing on the time decay of this solution due to dispersion. Finally, we extend these ideas to oscillatory integrals, proving the stationary phase theorem to demonstrate asymptotic decay in one spatial dimension.

Acknowledgments. This paper was completed as part of the Berkeley Mathematics Directed Reading Program. I thank my mentor Ning Tang for his guidance. His notes that formed the foundation of this work are listed in my references (4.2).

2. Foundations of Fourier Analysis

The Fourier transform converts a function of a spatial variable into a function of a frequency variable. This transformation is particularly useful in analyzing wave solutions, as frequency directly relates to properties such as velocity and energy. Additionally, the frequency domain often provides a simpler perspective on wave behavior. In fact, many operations are more straightforward in Fourier space.

In this section, we discuss relevant properties of the Fourier transform that will be used throughout this paper. To simplify the discussion, we will work with "nice" functions—those that decay rapidly to 0 as $|x| \to \infty$. This assumption allows us to neglect boundary terms when integrating by parts. Specifically, if at least one of the functions u or v is "nice", the formula

$$\int u dv = -\int v du$$

holds.

2.1. **Definition of the transform.** The Fourier transform maps a function $f(x)$ from the spatial domain to the frequency domain. It is defined as:

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx, \tag{2.1}$$

where x is the spatial variable and ξ is the frequency variable. By integrating $f(x)$ against the oscillatory exponential $e^{ix\xi}$, we measure how much of $f(x)$ correlates with a particular frequency ξ. This returns a function describing the relevant frequencies of $f(x)$.

The inverse Fourier transform recovers the original function $f(x)$ from its Fourier transform $\widehat{f}(\xi)$. It is defined as:

$$\mathcal{F}^{-1}(\widehat{f})(x) := f(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)e^{-ix\xi}d\xi. \tag{2.2}$$

This sums all frequency components $\widehat{f}(\xi)$ weighted with respect to the complex exponential $e^{ix\xi}$. This transformation is normalized by the constant $\frac{1}{2\pi}$.

2.2. **Properties in Fourier space.** We will now review some mathematical properties of Fourier transforms.

An inner product is an operation that takes in two vectors and returns a scalar value. We can define the inner product of functions f and g as:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx, \tag{2.3}$$

where $\overline{g(x)}$ is the complex conjugate of $g(x)$.

This inner product has an interesting property in the Fourier space:

Theorem 2.1. *For functions f and g,*

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \widehat{f}, \widehat{g}\rangle.$$

Proof. By definition of the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\alpha)e^{ix\alpha}d\alpha, \quad g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\beta)e^{ix\beta}d\beta.$$

Substitute these functions into the definition of the inner product to see

$$\langle f, g\rangle = \int_{\mathbb{R}} \left(\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\alpha)e^{ix\alpha}\, d\alpha\right)\overline{\left(\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\beta)e^{ix\beta}\, d\beta\right)}dx.$$

All integrals can be pulled to the front. Take the complex conjugate and simplify to

$$\langle f, g\rangle = \frac{1}{(2\pi)^2}\iiint_{\mathbb{R}} \widehat{f}(\alpha)\overline{\widehat{g}(\beta)}e^{ix(\alpha-\beta)}d\alpha d\beta dx.$$

Integrate the exponential term $e^{ix(\alpha-\beta)}$ with respect to x, giving the Dirac delta function $2\pi\delta(\alpha-\beta)$. This function collapses the integral over β, yielding:

$$\langle f, g\rangle = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}} \widehat{f}(\alpha)\overline{\widehat{g}(\beta)}2\pi\delta(\alpha-\beta)d\alpha d\beta = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\alpha)\overline{\widehat{g}(\alpha)}d\alpha.$$

Employing the definition of the inner product, we have shown

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \widehat{f}, \widehat{g}\rangle,$$

as desired. □

We will also examine the Fourier transform on Gaussian functions.

Theorem 2.2. *The Fourier transform of a Gaussian function $f(x) = e^{-\alpha\frac{1}{2}x^2}$ is given by:*

$$\mathcal{F}(f)(\xi) = \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{\xi^2}{2\alpha}},$$

where α is a constant.

Proof. Apply the definition of the Fourier transform:

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{-\alpha\frac{1}{2}x^2} e^{-ix\xi} dx.$$

Combine the two exponentials:

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{-\frac{\alpha}{2}(x^2 + \frac{2i\xi}{\alpha}x)} dx.$$

We can complete the square in the exponent:

$$\mathcal{F}(f)(\xi) = e^{\frac{\xi^2}{2\alpha}} \int_{\mathbb{R}} e^{-\frac{\alpha}{2}(x + \frac{i\xi}{\alpha})^2} dx.$$

This is now a standard Gaussian integral. The result of this expression is well-known:

$$\mathcal{F}(f)(\xi) = \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{\xi^2}{2\alpha}}.$$

□

3. The Wave Equation and Dispersion

As stated in the introduction, we will use the Fourier transform to solve the following 1-D wave equation:

$$\left(-\partial_t^2 + \partial_x^2\right) u = 0. \tag{3.1}$$

Physically, we expect the solution to be a superposition of left-moving and right-moving waves, and we will see how this result emerges naturally from Fourier analysis.

3.1. **Solution to the wave equation.** In order to simplify the wave equation, we apply the Fourier transform with respect to the spatial variable x. This transforms the wave equation to the frequency domain with respect to ξ, giving:

$$\partial_t^2 \widehat{u}(\xi, t) + \xi^2 \widehat{u}(\xi, t) = 0. \tag{3.2}$$

This transformed equation is a second-order ordinary differential equation (ODE) in the time variable t. The general solution to this ODE is:

$$\widehat{u}(\xi, t) = \widehat{u_+}(\xi) e^{it\xi} + \widehat{u_-}(\xi) e^{-it\xi},$$

where $\widehat{u_+}(\xi)$ and $\widehat{u_-}(\xi)$ are functions determined by the initial conditions of the problem.

To recover the solution $u(x, t)$ in the spatial domain, we simply apply the inverse Fourier transform (2.2) to $\widehat{u}(\xi, t)$. This yields:

$$u(x, t) = \int_{\mathbb{R}} \widehat{u_+}(\xi) e^{i(t\xi + x\xi)} \frac{\mathrm{d}\xi}{2\pi} + \int_{\mathbb{R}} \widehat{u_-}(\xi) e^{i(-t\xi + x\xi)} \frac{\mathrm{d}\xi}{2\pi}.$$

This solution reflects the physical expectation that waves travel in both directions. With this example, we see how the Fourier transform simplifies the wave equation, providing an elegant framework to solve a complex second-order PDE.

Now that we have a solution to the wave equation, we can investigate its decay as $|t| \to \infty$. Let us consider one of the integrals in isolation; because the solution is symmetric with respect to $u_+(\xi)$ and $u_-(\xi)$, showing decay for one function will imply decay in the other.

4. The Stationary Phase Theorem

In this section, we introduce two equations that estimate the value of an oscillatory integral of the form

$$I(\lambda) = \int_{\mathbb{R}} a(x) e^{i\lambda\varphi(x)} dx,$$

reminiscent of the wave solution

$$\int_{\mathbb{R}} \widehat{u_+}(\xi) e^{i(t\xi + x\xi)} \frac{\mathrm{d}\xi}{2\pi}.$$

To introduce the following theorems, we must first define the support of a function. The support of $a(x)$, denoted *suppa*, is the set of all points x where $a(x)$ is nonzero:

$$\mathit{suppa} := \{x : a(x) \neq 0\}.$$

A function is said to have compact support if *suppa* is a bounded subset of $\mathbb{R}$. An infinitely differentiable function $a(x)$ with compact support is notated as $a \in C_c^\infty(\mathbb{R})$, where the subscript c indicates compact support and the superscript ∞ indicates infinite differentiability.

These properties are important in the context of wave decay because compact support ensures that the function is nonzero on a finite interval, making analysis more manageable. Infinite differentiability ensures that we can use tools like integration by parts and asymptotic expansions.

We will also introduce the definition of a supremum. The supremum of a set is the smallest upper bound, meaning it is the smallest value greater than or equal to every element in the set. For example, the supremum of

$$A = \{x : 0 < x < 1\}$$

is 1. This is denoted

$$\sup_{x \in \mathbb{R}} A = 1.$$

This definition is often used in inequalities, and we will employ this concept in the following functions.

4.1. Non-stationary phase.

We begin by investigating the case in which the phase function $\varphi(x)$ is non-stationary, that is, $\varphi'(x) \neq 0$ for all x. The non-stationary phase theorem is stated in one spatial dimension as follows:

Theorem 4.1 (Non-stationary phase theorem)**.** *Suppose the phase function $\varphi(x)$ satisfies the following conditions:*

(1) All derivatives are uniformly bounded, meaning there exists a constant C_k dependent on the order k of the derivative such that

$$\sup_{x\in\mathbb{R}} |\varphi^{(k)}(x)| \leq C_k.$$

(2) There are no critical points of $\varphi(x)$, meaning the first derivative is nonzero everywhere on the support of $a(x)$:

$$|\varphi'(x)| > 0.$$

Then, for any integer N, there exists a constant $C_N = C(suppa, N, \varphi)$ such that the oscillatory integral $I(\lambda)$ decays with λ^{-N}:

$$|I(\lambda)| \leq C_N \sup_{|\alpha|\leq N, x\in\mathbb{R}} |\partial^\alpha a(x)|\, \lambda^{-N}.$$

This theorem is shown through repeated substitution of the identity

$$e^{i\lambda\varphi(x)} = \frac{1}{i\lambda\varphi'(x)}\partial_x\left(e^{i\lambda\varphi(x)}\right),$$

which allows us to pull out a factor of λ^{-N} for all $N \in \mathbb{Z}$. This result indicates that under non-stationary conditions, the oscillatory integral $I(\lambda)$ can decay arbitrarily fast. This makes physical sense; if the phase function is changing continuously, oscillations cancel each other out, leading to dramatic decay of the wave.

4.2. **Stationary phase.** However, we must now investigate the case in which $\varphi(x)$ contains a critical point. To address this, we first introduce the concept of a bump function, denoted $\chi(x)$. A bump function is a smooth, infinitely differentiable function defined on the interval (α, β) by:

$$\chi(x) = \begin{cases} 1 & x \in (\alpha,\beta) \\ 0 & else. \end{cases}$$

Note that

$$1-\chi(x) = \begin{cases} 0 & x \in (\alpha,\beta) \\ 1 & else. \end{cases}$$

This bump function is useful because it allows us to isolate a specific region of interest in an integral. We can split integrals over $\mathbb{R}$ into two parts to simplify calculations; specifically, we will use this function to apply the non-stationary phase theorem to regions outside the critical point.

Theorem 4.2 (Stationary phase theorem)**.** *Let $a(x)$ be an infinitely differentiable function with compact support, denoted $a \in C_c^\infty(\mathbb{R})$. Assume suppa is contained in the finite interval (α, β).*

Let the phase function $\varphi(x)$ be infinitely differentiable: $\varphi \in C^\infty(\mathbb{R};\mathbb{R})$. Furthermore, assume φ has a unique non-degenerate critical point x_0 in the interval (α, β). Specifically,

$$\varphi'(x_0) = 0, \varphi''(x_0) \neq 0.$$

Then, $I(\lambda)$ *decays with* $\lambda^{-\frac{1}{2}}$ *in the following expansion:*

$$I(\lambda) = e^{i\lambda\varphi(x_0)} \left(\frac{2\pi}{|\varphi''(x_0)|} \right)^{\frac{1}{2}} e^{\frac{i\pi}{4}\mathrm{sgn}\varphi''(x_0)} \lambda^{-\frac{1}{2}} \left(a_0 + a_1\lambda^{-1} + \cdots + a_{N-1}\lambda^{-N+1} \right) + \lambda^{-N-\frac{1}{2}} S_N,$$

where $a_0 = a(x_0)$ *and the constant* S_N *satisfies*

$$|S_N| \leq C_N \sup_{|\alpha| \leq 2N+2} |\partial^\alpha a| .$$

Proof. We first begin with the oscillatory integral

$$I(\lambda) = \int_{\mathbb{R}} a(x) e^{i\lambda\varphi(x)} dx.$$

Let $\chi(x) \in C_c^\infty(\mathbb{R})$ be a bump function localized to the critical point x_0. Multiply the integral $I(\lambda)$ by $1 = \chi(x) + (1 - \chi(x))$ to split the integral into two parts:

$$I(\lambda) = \int_{\mathbb{R}} \chi(x) a(x) e^{i\lambda\varphi(x)} dx + \int_{\mathbb{R}} (1 - \chi(x)) a(x) e^{i\lambda\varphi(x)} dx.$$

Notice the second integral does not contain the stationary point x_0, meaning we can apply the non-stationary phase theorem (Theorem 4.1). This term decays with λ^{-N}. We will define its coefficient S_N as given in the non-stationary phase theorem:

$$|S_N| \leq C_N \sup_{|\alpha| \leq N, x \in \mathbb{R}} |\partial^\alpha a(x)| .$$

Thus,

$$I(\lambda) = \int_{\mathbb{R}} \chi(x) a(x) e^{i\lambda\varphi(x)} dx + \lambda^{-N} S_N.$$

We will now compute the contribution from the first integral. First, Taylor expand $\varphi(x)$ around the critical point x_0 up to the quadratic term:

$$\varphi(x) \approx \varphi(x_0) + \varphi'(x_0)(x - x_0) + \frac{\varphi''(x_0)}{2}(x - x_0)^2$$

By definition, $\varphi'(x_0) = 0$, so the first-order term vanishes. To evaluate this integral, we define a new variable $y(x)$:

$$y(x) \approx |\varphi''(x_0)|^{\frac{1}{2}} (x - x_0)$$

where $y(x_0) = 0$ and $y'(x_0) = |\varphi''(x_0)|^{\frac{1}{2}}$. Let $\epsilon = \mathrm{sgn}(\varphi''(x_0))$. With this substitution,

$$\varphi(x) = \varphi(x_0) + \frac{1}{2}\epsilon(y(x))^2.$$

Substitute this into our first integral:

$$\int_{\mathbb{R}} \chi(x) a(x) e^{i\lambda\varphi(x)} dx = \int_{\mathbb{R}} \chi(x) a(x) e^{i\lambda(\varphi(x_0) + \frac{1}{2}\epsilon(y(x))^2)} dx$$

Factor out the constant phase term:

$$\int_{\mathbb{R}} \chi(x) a(x) e^{i\lambda\varphi(x)} dx = e^{i\lambda\varphi(x_0)} \int_{\mathbb{R}} \chi(x) a(x) e^{i\lambda\frac{1}{2}\epsilon(y(x))^2} dx.$$

Notice x is now a function of y, so we will make a change of variable from x to y. Change the integration variable and insert the Jacobian $|\frac{dx}{dy}| = |\varphi''(x_0)|^{-\frac{1}{2}}$:

$$\int_{\mathbb{R}} \chi(x)a(x)e^{i\lambda\varphi(x)}dx = e^{i\lambda\varphi(x_0)} \int_{\mathbb{R}} \chi(y(x))a(y(x))e^{i\lambda\frac{1}{2}\epsilon(y(x))^2}|\varphi''(x_0)|^{-\frac{1}{2}}dy.$$

For simplicity, define $b(y) = \chi(y(x))a(y(x))$. Pull the Jacobian in front:

$$\int_{\mathbb{R}} \chi(x)a(x)e^{i\lambda\varphi(x)}dx = e^{i\lambda\varphi(x_0)}|\varphi''(x_0)|^{-\frac{1}{2}} \int_{\mathbb{R}} b(y)e^{i\lambda\frac{1}{2}\epsilon y^2}dy$$

We now focus on the inner integral expression. By the definition of the inner product, we can rewrite this as

$$\int_{\mathbb{R}} b(y)e^{i\lambda\frac{1}{2}\epsilon y^2}dy = \langle b, e^{i\lambda\frac{1}{2}\epsilon y^2}\rangle.$$

Employ Theorem 2.1 to yield:

$$\langle b, e^{i\lambda\frac{1}{2}\epsilon y^2}\rangle = \frac{1}{2\pi}\langle \widehat{b}, \widehat{e^{i\lambda\frac{1}{2}\epsilon y^2}}\rangle.$$

Notice that the second entry of the inner product is the Fourier transform of a Gaussian. We can apply Theorem 2.2:

$$\widehat{e^{i\lambda\frac{1}{2}\epsilon y^2}} = \sqrt{\frac{2\pi}{\lambda}}e^{\frac{i\pi}{4}\epsilon}e^{\frac{i\epsilon\xi^2}{2\lambda}}.$$

Substitute this into the definition of the inner product:

$$\int_{\mathbb{R}} b(y)e^{i\lambda\frac{1}{2}\epsilon y^2}dy = \frac{1}{2\pi}\sqrt{\frac{2\pi}{\lambda}}e^{\frac{i\pi}{4}\epsilon}\int_{\mathbb{R}} \widehat{b}(\xi)e^{-i\frac{\epsilon\xi^2}{2\lambda}}d\xi = \frac{1}{\sqrt{2\pi\lambda}}e^{\frac{i\pi}{4}\epsilon}\int_{\mathbb{R}} \widehat{b}(\xi)e^{-i\frac{\epsilon\xi^2}{2\lambda}}d\xi.$$

Plug this back into the expression for $I(\lambda)$ to yield

$$I(\lambda) = e^{i\lambda\varphi(x_0)}\frac{1}{\sqrt{2\pi\lambda}}e^{\frac{i\pi}{4}\epsilon}\int_{\mathbb{R}} \widehat{b}(\xi)e^{-i\frac{\epsilon\xi^2}{2\lambda}} + \lambda^{-N}S_N.$$

To arrive at the final form of the expression, we Taylor expand $e^{-i\frac{\epsilon\xi^2}{2\lambda}}$ around $\xi = 0$:

$$e^{-i\frac{\epsilon\xi^2}{2\lambda}} \approx 1 - i\frac{\epsilon}{2\lambda}\xi^2 - \frac{\epsilon^2}{8\lambda^2}\xi^4...$$

Notice that the terms decay with corresponding powers of λ.

This expansion pulls out factors of ξ^k. By properties of the Fourier transform,

$$\int_{\mathbb{R}} \xi^k\widehat{b}(\xi)d\xi = \int_{\mathbb{R}} \xi^k\widehat{b}(\xi)e^{i0\xi}d\xi = \widehat{\xi^k\widehat{b}(\xi)}(0) = \partial^k b(0).$$

In this way, we can express $b^{(k)}(0)$ in terms of the derivatives of $a(x)$ at x_0. Absorb the Nth derivative into the boundary term. Finally, we write

$$I(\lambda) = e^{i\lambda\varphi(x_0)}\left(\frac{2\pi}{|\varphi''(x_0)|}\right)^{\frac{1}{2}}e^{\frac{i\pi}{4}\mathrm{sgn}\varphi''(x_0)}\lambda^{-\frac{1}{2}}\left(a_0 + a_1\lambda^{-1} + \cdots + a_{N-1}\lambda^{-N+1}\right) + \lambda^{-N-\frac{1}{2}}S_N,$$

where $a_0, ...a_{N-1}$ depend on the derivatives of $a(x)$ at x_0, each decaying with the corresponding power of λ^{-k}. This completes the proof. □

Ultimately, we have shown that the stationary point x_0 of $\varphi(x)$ has the greatest contribution to the value of $I(\lambda)$ as $\lambda \to \infty$.

References

[Oh18] S-J. Oh. *Variations on a Theme: On the Dispersion of Waves.* 2018. https://math.berkeley.edu/~sjoh/pdfs/dispersion.pdf.

[Tan24] N. Tang. *Directed Reading Program - Fall 2024.* 2024.

University of California, Berkeley, CA 94720

Email address: laneysaye@berkeley.edu