The Vershik-Okounkov Approach to the Representation of S_n

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Contents

1 Introduction

This paper presents an overview of the work of A. Vershik and A. Okounkov on the Representation Theory of the Symmetric Groups. Specifically focusing on their results showing how the natural chain formed by the symmetric groups ($S_{n+1} \supset S_n \supset \cdots$) is reflected in their representations. Furthermore, discussing how the combinatorics of Young diagrams and Young tableaux can be described as emerging from the relationships among the symmetric groups rather than constructed as an external framework.

1.1 Representation Theory

This section will present definitions, theorems, and concepts that A. Vershik and A. Okounkov work is based on. We will be focused on finite groups in this paper thus definitions and theorems will be modified accordingly, additionally in this paper we will work over $\mathbb{C}$.

Definition 1.1. A **representation** of a finite group G is a vector space V over a filed $\mathbb{C}$ equipped with the homomorphism $\rho : G \to GL(V)$.

Remark 1.1. From now on we will refer to the representation of a finite group G as (ρ, V), where ρ is the homomorphism described in *definition 1.1* and elements of such representation will be denoted as $\rho(g)$.

In relation to the work that will be focused on this paper the following topics of representation theory is of the most importance.

Definition 1.2. A **sub-representation** of a representation (ρ, V) is a subspace $U \subset V$ which is invariant under operations of $\rho(g)$, $g \in G$.

Definition 1.3. An **irreducible** representation $(\rho, V), \neq 0$, is a representation such that it's only sub-representations are 0 and itself.

Let us now define how we relate two representations, take to be $(\rho, V_1), (\pi, V_2)$, of a group G. A **homomorphism** of representation is a map $\phi : V_1, \to V_2$ such that $\phi(\rho(g)v) = \pi(g)(\phi(v))$ for $g \in G, v \in V_1$. We call ϕ

an **isomorphism** of representations if it is an isomorphism of vector spaces. We define the set of homomorphism from V_1 to V_2 as $\mathrm{Hom}_G(V_1, V_2)$.

Remark 1.2. Homomorphism of representations are also referred to as **intertwining maps**.

We then result with the one of the most important theorems of representation theory that will be referred to extensively in the paper.

Theorem 1.1. Schur's Lemma ([2], Proposition 2.3.9): Let V be a finite dimensional irreducible representation of an algebra A over $\mathbb{C}$ (an algebraically closed field), and let $\phi : V \to V$ be an intertwining operator. Then $\phi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$ (a scalar operator).

Proof. Let $\lambda \in \mathbb{C}$ be a root of ϕ's characteristic polynomial, thus an eigenvalue of ϕ. Hence take $\phi - \lambda \cdot \mathrm{Id}$ to be an intertwining operator, i.e, $\phi - \lambda \cdot \mathrm{Id} : V \to V$. Since λ is an eigenvalue of ϕ, we have $\det(\phi - \lambda \cdot \mathrm{Id}) = 0$, so $\phi - \lambda \cdot \mathrm{Id}$ is not invertible. This implies that $\ker(\phi - \lambda \cdot \mathrm{Id}) \neq \{0\}$ thus $\exists v \in V : (\phi - \lambda \cdot \mathrm{Id})(v) = 0$, or rather, $\phi(v) = \lambda \cdot \mathrm{Id}(v)$. Although, by the irreducibility of V the invariant subspace $\ker(\phi - \lambda \cdot \mathrm{Id})$ of V must be all of V which implies $\phi - \lambda \cdot \mathrm{Id} = 0$ as an operator $\implies \phi = \lambda \cdot \mathrm{Id}$. $\square$

The following corollary of Schur's Lemma then gives the basis to our paper.

Corollary 1.1.1. Let A be a commutative algebra. Then every irreducible finite dimensional representation V of A is 1-dimensional.

Proof. The inclusion to the left is obvious, that is if we have a 1-dimensional finite representation V then it's only subspaces are $\{0\}$ and itself, thus irreducible. For the inclusion to the right:
Take V to be an irreducible representation and consider the intertwining operator $\rho(a) : V \to V$ for $a \in A$, then $\rho(a) = \lambda \cdot \mathrm{Id}$ for $\lambda \in \mathbb{C}$ by Theorem 1.1, that is $\rho(a)$ acts as a scalar operator on V. Hence take a subspace of V, W, then $\rho(a) \cdot W = \lambda \cdot W \in W$ as W is closed under scalar operations. So we have an invariant subspace under $\rho(a)$ making W a subrepresentation of V. Although since we have V to be irreducible then $W = \{0\}$ or $V \implies \dim V = 1$. $\square$

A result of representation theory is that of **semi-simple group**, a group such that it's representation V is a direct sum of irreducible representations. Let us define it as

$$V = V_1^{\oplus \alpha_1} \oplus \cdots \oplus V_i^{\oplus \alpha_i}$$

where V_i are irreducible, pairwise non-isomorphic, representations and α_i are the multiplicities of V_i in V.

Theorem 1.2. ([2], Proposition 3.5.8): Any finite dimensional representation of a finite group G is completely reducible (that is, isomorphic to a direct sum of irreducible representations).

1.1.1 The Symmetric Group S_n

The symmetric group S_n at the core of this paper and is one of the most interesting groups to work with. The following are summarizations of key definitions and properties of S_n:

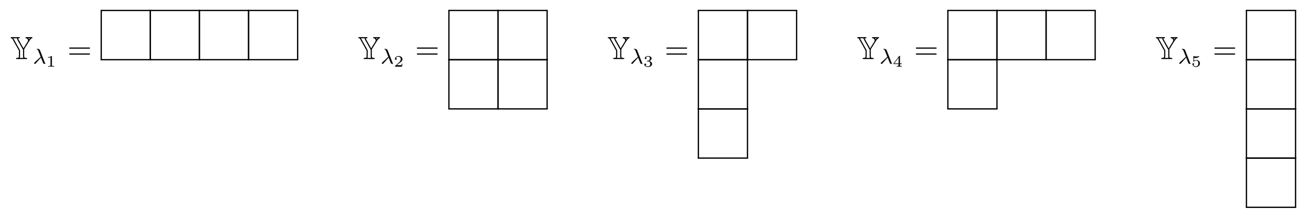
Definition 1.4. The **symmetric group**,S_n, is the group of all permutations of the set $\{1, 2, \dots, n\}$ with order $n!$.

The structure of S_n is crucial for understanding its representations, and its subgroup relationships reveal deeper combinatorial connections. For instance, we have a natural inclusion of $S_{n-1} \subset S_n$ as $\{\sigma \in S_n | \sigma(n) = n\} \simeq S_{n-1}$. A **partition** λ of n is a representation of n in the form $n = \lambda_1 + \cdots + \lambda_p$, where λ_i are positive integers and $\lambda_i \geq \lambda_{i+1}$. [2]

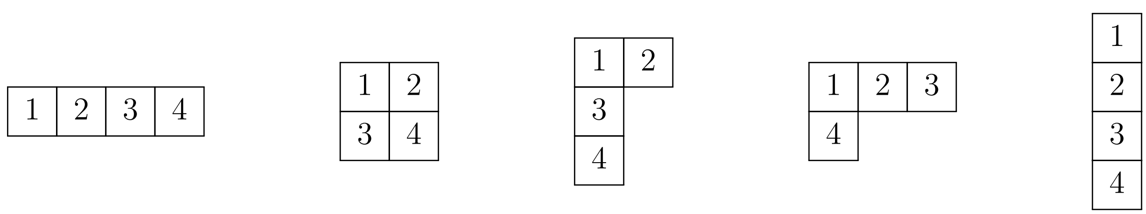
Definition 1.5. A **Young diagram** is a graphical representation of the partition of n. That is for λ we attach a Young diagram which is the union of rectangles $-j \leq y \leq -j+1, 0 \leq c \leq \lambda_j$ in the coordinate plane for $j = 1, \ldots, p$. Let us denote it with $\mathbb{Y}_n$ [2]

A **Young tableau** corresponding to a Young diagram is the result of filling the numbers $1, \ldots, n$ into the squares of the diagram in some way without repetition.

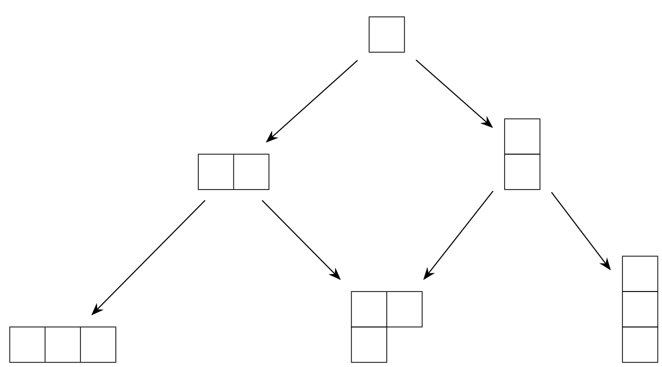
Take S_4, for example, and note that the partitions of 4 correspond to the distinct conjugacy classes of S_4. Then it's young diagram, $\mathbb{Y}_4$, will have the following structure corresponding to it's partitions $\lambda_1 = (4)$ $\lambda_2 = (2,2)$ $\lambda_3 = (2,1,1)$ $\lambda_4 = (3,1)$ $\lambda_5 = (1,1,1,1)$



Then the Young tableau will be



The item of interest in this paper will be the **Young Graph**, which is a branching graph for Young diagrams of S_n, in particular.



Where the first level corresponds to S_1, the second S_2, and the third S_3, and so on.

1.1.2 Characters

One of, if not the most important tool in representation theory is **character theory**. It is a tool based on the idea that knowledge of the eigenvalues of each element of G is enough to sufficiently describe the representation of such group.

A **character**, χ_V, of a representation V of G is the complex-valued function on the group defined as:

$$\chi_V(g) = \text{Tr}(g|_V)$$

the trace of g on V. Characters also satisfy the following for representations V, W of G

$$\begin{aligned}\chi_{V\oplus W}(g) &= \chi_V(g) + \chi_W(g)\\ \chi_{V\otimes W}(g) &= \chi_V(g) \cdot \chi_W(g)\\ \chi_{V^*}(g) &= \overline{\chi_V(g)}\end{aligned}$$

where V^* is the dual of V. [3] Consider the set $V^G = \{v \in V : gv = v, \forall g \in G\}$, thus we wonder how we can find V^G explicitly. Thus we turn to the formula that takes the average of all endomorphisms $g : V \to V$, that is

$$\varphi = \frac{1}{|G|}\sum_{g\in G} g \in \text{End}(V)$$

Then if we only want to know the number m of copies of the trivial representation appearing in the decomposition of V, we simply look at the **trace** of φ that is

$$\begin{aligned}m = \dim V^G &= \text{Trace}(\varphi)\\ &= \frac{1}{|G|}\sum_{g\in G}\text{Trace}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\end{aligned}$$

From here the following is then developed [3]

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g) = \begin{cases}1 \text{ if } V \simeq W\\ 0 \text{ if } V \not\simeq W\end{cases}$$

Where we deduce that

Theorem 1.3. The characters of the irreducible representations of G are orthonormal.

In consequence to the above theorem, we get that the number of irreducible representations of G is less than or equal to the number of conjugacy classes, much to do with how the character of a representation of a group G is really a "function on the set of conjugacy classes in G" [3], and in working with S_n we find that it is equal. To see these principles in action, let us construct the character table of S_4.

Recall the conjugacy classes of S_4, that is $\{(1), (12), (123), (1234), (12)(34)\}$, which correspond to the partitions of 4 (see Section 1.1.1). To begin, take the trivial representation (U), which when acted on the conjugacy classes has values $(1, 1, 1, 1, 1)$. Then when we take the alternating representation (U'), that is the representation $\rho(\sigma) = \text{sgn}(\sigma) = \begin{cases}1 \text{ if } \sigma \text{ is an even permutation}\\ -1 \text{ if } \sigma \text{ is an odd permutation}\end{cases}$ where $\sigma \in S_4$ is even if it's decomposition into transpositions has an even amount of 2-cycles, and odd otherwise. The alternating representation will have values/characters $(1, -1, 1, -1, 1)$. Now let us consider the standard representation of S_4, that is the representation of S_4 on the vector space $V = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$ where we have the following decomposition

$$\mathbb{C}^4 = V \oplus U$$

In order to find the values of the characters of V, we must find the characters of $\mathbb{C}^4$ and use our property listed previously, in particular $\chi_{V \oplus W} = \chi_V + \chi_W$. Thus let us look at $\mathbb{C}^4$ under the permutation actions of the conjugacy classes of S_4, that is the **fixed point formula**.

Definition 1.6. The **fixed point formula** states that if V is the permutation representation associated to the action of a group G on a finite set X, then $\chi_V(g)$ is the number of elements of X fixed by g.

Take the standard vectors of $\mathbb{C}^4$ to be $\{\alpha_1, \alpha_2, \alpha_3\, \alpha_4\}$ then

$$\begin{aligned} (1) &\text{ fixes 4 elements} \\ (12) &\text{ fixes 2 elements} \\ (123) &\text{ fixes 1 elements} \\ (1234) &\text{ fixes 0 elements} \\ (12)(34) &\text{ fixes 0 elements} \end{aligned}$$

Thus $\chi_{\mathbb{C}^4} = (4, 2, 1, 0, 0)$, so now we can solve $\chi_{\mathbb{C}^4} = \chi_V + \chi_U \longrightarrow \chi_V = \chi_{\mathbb{C}^4} - \chi_U$ thus we result in $\chi_V = (3, 1, 0, -1, -1)$

To get the remaining two representations, we will first tensor our standard representation with our alternating representation, then construct our last representation through the orthogonality relations of characters from Theorem 1.3.

Thus we will have $V' = V \otimes U'$ which then gets us $\chi_{V'} = \chi_U \times \chi_{U'} = (3, 1, 0, -1, -1) \times (1, -1, 1, -1, 1) = (3, -1, 0, 1, -1)$. Then from our orthogonality relations we obtain the last row and get the following character table.

S_4	(1)	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U’	1	-1	1	-1	1
V	3	1	0	-1	-1
V’	3	-1	0	1	-1
W	2	0	-1	0	2

1.1.3 The Group Algebra $\mathbb{C}[G]$

Studying S_n can get quite abstract and complex once we reach higher values of n. Thus we introduce the group algebra $\mathbb{C}[G]$ (where in this paper we take $G = S_n$) which will simplify and make studying S_n more computational rather than abstract. Let us list key definitions and properties that will be used in this paper.

Definition 1.7. The $\mathbb{C}[G]$, of a finite group G is the vector space over $\mathbb{C}$ with basis elements $\{g : g \in G\}$, that is

$$\mathbb{C}[G] = \{ag : a \in \mathbb{C}, g \in G\}$$

The center of the group algebra $\mathbb{C}[G]$, $A(\mathbb{C}[G])$, is the subalgebra of elements that commute with all others in $\mathbb{C}[G]$. Of most importance, the elemtns of $A(\mathbb{C}[G])$ are linear combinations of sums of group elements in the same conjugacy class. A key result from studying $\mathbb{C}[G]$ is the following

Theorem 1.4. Wedderburn-Artin Theorem: If R is a semi-prime left arinian ring then

$$R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where each D_i is a division ring and $M_n(D)$ denotes the ring of $n \times n$ matrices over D.

In relation to group algebras, which are semi-prime left arinian rings, we have the following decomposition that plays a huge role in A. Vershik and A. Okounkov work

$$\mathbb{C}[G] \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$$

where n_i is the dimension of the i-th irreducible representation.

2 Motive

The representation theory of symmetric groups has long been a cornerstone of algebra and combinatorics, providing deep insights into the structure and relationships of these fundamental groups. A classical approach to studying representations of symmetric groups relies heavily on the combinatorial framework of Young diagrams and tableaux, which encode the structure of irreducible representations. However, this approach often introduces these combinatorial tools as external constructions, leaving their origin within the theory itself somewhat opaque. The work of Vershik and Okounkov offers a novel perspective by showing that these combinatorial objects naturally arise from the inherent relationships among symmetric groups through the chain of inclusions $S_{n+1} \supset S_n \supset \cdots$. This perspective not only unifies the combinatorial and algebraic viewpoints but also provides a more intrinsic understanding of the representations of symmetric groups.

The motive behind this approach lies in simplifying and systematizing the study of symmetric group representations by leveraging the natural structure of the symmetric group chain. By analyzing how representations of S_n extend or restrict to representations of S_{n+1} and vice versa, Vershik and Okounkov uncovered an elegant framework where combinatorial structures such as Young diagrams emerge directly from these transitions. This method avoids the need to impose external combinatorial tools, instead deriving them as a consequence of the group relationships. This intrinsic viewpoint not only provides new insights into symmetric groups but also extends to broader applications, bridging representation theory, algebraic combinatorics, and probability theory in innovative ways.

3 Methods

Now we will get into the actual result Vershik and Okounkov have gotten to, that is the isomorphism between Bratteli Diagram of the symmetric groups and the Young diagram.

$$\begin{tikzcd}
 & & U_1 \arrow[dl] \arrow[dr] & & \\
 & U_2 \arrow[dl] \arrow[dr] & & U_2' \arrow[dl] \arrow[dr] & \\
U_3 & & V & & U_3'
\end{tikzcd}$$

Figure 1: Bratteli diagram of S_3 [1]

3.1 Bratteli Diagrams

Definition 3.1. A **Bratteli diagram**, also called a **branching graph**, of a chain of finite groups $\{1\} = G_0 \subset G_1 \subset G_2 \subset \cdots$ is a graph that's vertices are elements of the set

$$\bigcup_{n \geq 0} G(n)^\wedge$$

Where $G(n)^\wedge$ is the set of isomorphism classes of complex irreducible representation of $G(n)$, and between irreducible representation μ of G_n and λ of G_{n+1} there are k directed edges from μ to λ if k is the multiplicity of μ in the restriction of λ to G_n. [4]

The Bratteli diagram of S_n, relating to it's infinite chain of groups $\{1\} = S_1 \subset S_2 \subset \cdots$, has the structure shown in Figure 1. The vertices in the graph represent the one-dimensional irreducible representations of S_n where each level represents a different value of n, that is the first level corresponds to S_1 and the second level corresponds to S_2 and so on.

In particular we have U_1 to be the trivial representation, which corresponds to S_1's unique conjugacy class (see Section 1 for a brief overview of how this is derived). Then for S_2 it has two one-dimensional irreducible representations, corresponding to it's two conjugacy classes, and we find those two to be the trivial representation U_2 and the alternating representation U_2'. For S_3 we have similar results with the trivial representation U_3 and it's alternative representation U_3', but then we also get the standard representation V.

The arrows between these one-dimensional irreducible representations describe how representations of S_n restrict to S_{n-1}, or equivalently, how representations of S_{n-1} can be extended to S_n. Focusing on the extension of S_{n-1}, take $U_\lambda \in (S_{n-1})^\wedge$ and $U_\mu \in (S_n)^\wedge$ then we take U_λ and induce it up to S_n, and we find that V_μ will appear as one of the irreducible components of the induced representation:

$$\mathrm{Ind}_{S_{n-1}}^{S_n}(V_\lambda) = \bigoplus_{\lambda \to \mu} V_\mu$$

Now to relate this to the Young diagrams of S_n we will need the following.

3.2 Gelfand-Tsetlin Algebra and GZ Basis

To establish the isomorphism between the Bratteli diagram of the symmetric group S_n and the Young diagram, we introduce the **Gelfand-Tsetlin algebra** (GZ algebra) and the associated GZ basis. These concepts simplify the process of understanding the branching graph of S_n by providing a concrete algebraic framework. Working with the GZ algebra enables us to better analyze the relationship between representations of S_n and S_{n-1}.

Definition 3.2. The **Gelfand-Tsetlin algebra** $GZ(n)$ associated with S_n is the commutative subalgebra of $\mathbb{C}[S_n]$ generated by the centers $Z(\mathbb{C}[S_k])$ for $k = 1, 2, \ldots, n$.

Remark 3.1. The algebra $GZ(n)$ organizes the structure of irreducible representations in a way that aligns naturally with the branching rules of S_n to S_{n-1}.

The GZ algebra plays a crucial role in constructing the **GZ basis**, a basis for the irreducible representations of S_n that respects the hierarchical structure of the symmetric groups. The GZ basis enables us to work concretely with the representation theory of S_n by relating it to the branching graph.

3.2.1 GZ Basis and Branching Graph Simplicity

The GZ basis is constructed recursively, reflecting the restriction of representations from S_n to S_{n-1}. Let V_λ be an irreducible representation of S_n corresponding to a Young diagram λ. The GZ basis for V_λ is indexed by

$$\{\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\},$$

where each $\lambda^{(k)}$ represents a partition obtained by successively restricting λ from S_n to S_{n-1}, S_{n-2}, and so on.

Definition 3.3. A Bratteli diagram is **simple** if, in the diagram, there is either 0 or 1 arrow going between an irreducible S_n module and an S_{n-1} module.

To analyze the simplicity of the entire Bratteli diagram of S_n, we focus on the commutative properties of the GZ algebra. Specifically, we investigate the relationships between the centers $Z(\mathbb{C}[S_n])$ and $Z(\mathbb{C}[S_{n-1}])$. Define:

$$Z(n-1, n) \cong Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]).$$

This is given by the following

Theorem 3.1. **([4], Proposition 1.4)** The following two conditions are equivalent

1. The restriction of any finite dimensional irreducible complex representation of the algebra M to N has simple multiplicities.
2. The centralizer $Z(M, N)$ is commutative

Proof. **(1) ⇒ (2):** Assume that the restriction of irreducible representations of M to N has simple multiplicities. Consider the space $\mathrm{Hom}_N(V_\mu, V_\lambda)$, where V_μ and V_λ are irreducible representations of N and M, respectively. Since the multiplicities are simple, $\mathrm{Hom}_N(V_\mu, V_\lambda)$ must be one-dimensional. The space $\mathrm{Hom}_N(V_\mu, V_\lambda)$ is also a module for the centralizer $Z(M, N)$. If $Z(M, N)$ were not commutative, this module could not be one-dimensional. Therefore, $Z(M, N)$ must be commutative.

(2) ⇒ (1): Now, assume that $Z(M, N)$ is commutative. Since $Z(M, N)$ is commutative, any irreducible module for it is one-dimensional $\implies$ $\mathrm{Hom}_N(V_\mu, V_\lambda)$ is one-dimensional $\implies$ V_μ appears in V_λ with multiplicity 1 when restricted to N.

Therefore, the two statements are equivalent. $\square$

Showing that $Z(n-1,n)$ is commutative ensures that the branching graph of $\mathbb{C}[S_n]$ is simple, meaning that each irreducible representation of S_n corresponds uniquely to a path in the Bratteli diagram.

3.2.2 Connection to YJM Elements

To demonstrate commutativity, we leverage the **Young-Jucys-Murphy elements** $X_i \in \mathbb{C}[S_n]$ for $i = 1, 2, \ldots, n$, defined as:

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\,i),$$

We then see the following relation ([4])

$$X_i = \sum_1^j (j-1\,j) - \sum_i^k (k-1\,k)$$

for $(j-1\,j) \in S_j$ and $(k-1\,k) \in S_k$, in particular we get that X_i is the difference of an element of $Z(i)$ and an element of $Z(i-1) \implies X_i \in GZ(n)$. And we have stated $GZ(n)$, the Gelfand-Tsetlin algebra, is commutative thus the YJM elements commute.

Now we want to find some sort of relation between these commutative elements and that of the commutative sub-algebra of $\mathbb{C}(S_n)$, we then get the following

Theorem 3.2. ([4], Theorem 2.5) In the algebra $\mathbb{C}(S_n)$, consider it's center $Z(n)$ and the center $Z(n-1)$ of the sub-algebra $\mathbb{C}(S_{n-1})$ of $\mathbb{C}(S_n)$. Then

$$Z(n) \subset \langle Z(n-1), X_n \rangle$$

Proof. Expanding the Jucys-Murphy element X_n to X_n^2, we find:

$$X_n^2 = \sum_{i,j=1}^{n-1} (i,n)(j,n) = \sum_{i \neq j} (i,j,n) + (n-1)I,$$

Where $(i,j,n) \in S_n$ and $(n-1) \in Z(n-1)$. Thus, the element $\sum_{i \neq j}(i,j,n)$ lies in $\langle Z(n-1), X_n \rangle$.

Then we apply induction to generalize this process to cycles of length $k+1$. By induction, assume that conjugacy class indicators for cycles of length k are in $\langle Z(n-1), X_n \rangle$. Consider the element:

$$X_n \cdot \sum_{i_1,\ldots,i_{k-1}} (i_1, \ldots, i_{k-1}, n),$$

where X_n acts to extend each cycle of length k to a cycle of length $k+1$. This produces:

$$\sum_{i \neq i_s} (i,n)(i_1, \ldots, i_{k-1}, n) + \sum_{i, i_1, \ldots, i_k} (i, i_1, \ldots, i_k, n).$$

The first summand involves products of cycles, with terms that lie in $Z(n-1)$. The second summand gives the conjugacy class of cycles of length $k+1$ in S_n, which also lies in $\langle Z(n-1), X_n \rangle$.

Repeating this process, we conclude that all conjugacy class indicators for S_n, including those corresponding to cycles of any length, are in $\langle Z(n-1), X_n \rangle$.

Finally, the classical result states that the center $Z(n)$ of the group algebra $\mathbb{C}[S_n]$ is generated by the conjugacy class indicators, which correspond to all one-cycle permutations in S_n. Since we have shown that these are in $\langle Z(n-1), X_n \rangle$, it follows that:

$$Z(n) \subset \langle Z(n-1), X_n \rangle.$$

□

restricting this further we result with

Theorem 3.3. The centralizer $Z(n-1,1) \cong Z(\mathbb{C}(S_n), \mathbb{C}(S_{n-1}))$ of the algebra $\mathbb{C}(S_{n-1})$ in $\mathbb{C}(S_n)$ is generated by the center $Z(n-1)$ of $\mathbb{C}(S_{n-1})$ and the element X_n:

$$Z(n-1,1) = \langle Z(n-1), X_n \rangle$$

Proof. A linear basis for the centralizer $Z(n-1,1)$ is the union of a linear basis for $Z(n-1)$ and classes of the form:

$$\sum (i_1^{(1)}, \ldots, i_{k_1-1}^{(1)}, n)(i_1^{(2)}, \ldots, i_{k_2}^{(2)}) \ldots (i_1^{(3)}, \ldots, i_{k_3}^{(3)}),$$

for $i \in 1, \ldots, n-1$

Now, if we add these classes to the classes from $Z(n-1)$, as done previously, we obtain all the classes from $Z(n)$. Therefore, we can express the basis for $Z(n-1,1)$ as a linear combination of elements from the bases of $Z(n-1)$ and $Z(n)$:

$$Z(n-1,1) \subset \langle Z(n-1), Z(n) \rangle.$$

Finally, since $Z(n) \subset \langle Z(n-1), X_n \rangle$ (by Theorem 3.2) the result follows. □

And finally we remain with

Theorem 3.4. ([4], Theorem 2.9) The branching of the chain $\mathbb{C}(S_1) \subset \cdots \mathbb{C}(S_n)$ is simple.

Proof. We have shown that $\langle Z(n-1), X_n \rangle$ is commutative thus since $Z(n-1,1) \subset \langle Z(n-1), X_n \rangle$, $Z(n-1,1)$ is also commutative and from Theorem 3.1 the result follows. □

Verifying the commutativity of $\langle Z(\mathbb{C}[S_{n-1}]), X_n \rangle$ establishes the simplicity of the branching graph for $\mathbb{C}[S_n]$.

3.2.3 From Simplicity to Young Diagrams

The simplicity of the branching graph implies that each irreducible representation of S_n corresponds uniquely to a path in the branching graph, ensuring a one-to-one correspondence between these paths and irreducible representations. This alignment directly reflects the structure of the Young graph, where each node represents a partition, and edges correspond to valid branching steps.

By demonstrating simplicity, we establish that the combinatorial structure of the Young graph aligns perfectly with the algebraic branching graph of S_n. Furthermore, the spectrum of the Gelfand-Tsetlin algebra $GZ(n)$, which indexes irreducible representations, matches the content of the Young diagrams. This correspondence guarantees that every irreducible representation of S_n is uniquely labeled by a path in the Young graph, completing the isomorphism.

To solidify this connection, we note that the simplicity of the branching graph suffices to establish the connection between S_n and the Young graph, ensuring that $\text{Spec}(n) = \text{Cont}(n)$. Where $\text{Spec}(n)$ is the set of eigenvalues that arise when $GZ(n)$ acts on an irreducible representation of S_n and $\text{Cont}(n)$ is the content of a Young diagram of n boxes which uniquely characterizes the Young diagram and the associated irreducible representation.

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