The Probabilistic Method

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1 The Basic Method

1.1 The Probabilistic Method

Definition 1.1. *The Ramsey number $R(k, l)$ is the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k or there is a blue K_l.*

Proposition 1.2. *If $\binom{n}{k} * 2^{1-\binom{k}{2}} < 1$, then $R(k, k) > n$. Thus $R(k, k) > \lfloor 2^{k/2} \rfloor$ for all $k \geq 3$.*

Proof: Consider a random two-coloring of the edges of K_n obtained by coloring each edge independently either red or blue, where each color is equally likely. For any fixed set R of k vertices, let A_R be the event that the induced subgraph of K_n on R is monochromatic. Clearly, $Pr[A_R] = 2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$ possible choices for R, the probability that at least one of the events A_R occurs is at most $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Thus, with a positive probability, no event A_R occurs and there is a two-coloring of K_n without a monochromatic K_k, that is $R(k, k) > n$. Since $k \geq 3$ and we are taking $n = \lfloor 2^{k/2} \rfloor$, then $\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{2^{1+\frac{k}{2}}}{k!} \frac{n^k}{2^{k^2/2}} < 1$. Therefore $R(k, k) > \lfloor 2^{k/2} \rfloor$ for all $k \geq 3$. ■

1.2 Graph Theory

Definition 1.3. *A tournament on a set V of n players is an orientation $T = (V, E)$ of the edges of the complete graph on the set of vertices V. Thus for every two distinct elements $x, y \in V$, either $(x, y) or (y, x) \in E$ but not both.*
We also say that a tournament has a property S_k if for every set of k players, there is one that beats them all.

Theorem 1.4. *If $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k*

Proof: Consider a random tournament on the set $V = \{1, ..., \}$. For every fixed subset K of size k of V, let A_K be the event that there is no vertex that beats all the members of K. Clearly $Pr[A_K] = (1 - 2^{-k})^{n-k}$. This is because for every fixed vertex $v \in V - K$, the probability that v does not beat all the members of K is $1 - 2^{-k}$, and all these $n - k$ events corresponding to the various possible choices of v are independent. Thus is follows that $Pr[A_K foreveryK subsetofV] \leq \sum_{K \subseteq V} Pr[A_K] = \binom{n}{k}(1 - 2^{-k})^{n-k} < 1$. Therefore with positive probability, no event A_K occurs, that is, there is a tournament on n vertices that has the property S_k. ■

Definition 1.5. *A dominating set is a subset of vertices of a graph where every vertex is included in the set or is adjacent to a vertex within the set.*

Theorem 1.6. *Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of at most n $\frac{1+ln(\delta+1)}{\delta+1}$ vertices.*

Proof: Let $p \in [0, 1]$ be, for the moment arbitrary. Let us pick randomly and independently each vertex of V with probability p. Let X be the random set of all vertices picked at let $Y = Y_X$ be the random set of all vertices in $V - X$ that do not have any neighbor in X. The expected value of $|X|$ is clearly np. For each fixed vertex $v \in V$, $Pr[v \in Y] = Pr[vanditsneighborsarenotinX] \leq (1 - p)^{\delta+1}$. Since the expected value of a sum of random variables is the sum of their expectations and since the random variable $|Y|$ can be written as a sum of n indicator random variables X_v where $X_v = 1$ if $v \in Y$ and 0 otherwise, we conclude that the expected value of $|X| + |Y|$ is at most $np + n(1-p)^{\delta+1}$. Consequently, here is at least one choice of $X \subseteq V$ such that $|X| + |Y_X| \leq np + n(1-p)^{\delta+1}$. The set $U = X \cup Y_X$ is clearly a dominating set of G whose cardinality is at most this size. The above argument works for any arbitrary p in the above interval. We shall now optimize the result using calculus to bound $1 - p \leq e^{-p}$ to give the simpler bound $|U| \leq np + ne^{-p(\delta+1)}$. Take the derivative of the right side with respect to p and set it equal to zero. The right hand side is minimized at $p = \frac{ln(\delta+1)}{\delta+1}$. Thus we can then set p equal to this value and have now shown that $|U| \leq n\frac{1+ln(\delta+1)}{\delta+1}$ ■

Definition 1.7. *A cut in a graph $G = (V, E)$ is a partition of the set of vertices V into two nonempty disjoint sets $V = V_1 \cup V_2$. If $v_1 \in V_1$ and $v_2 \in V_2$, we say that the cut separates v_1, v_2. The size of the cut is number of edges of G having one end in V_1 and the other in V_2. In fact, we sometimes identify the cut with the set of these edges. The edge connectivity of G is the minimum size of a cut of G.*

Lemma 1.8. *Let $G = (V, E)$ be a graph with minimum degree δ, and let $V = V_1 \cup V_2$ be a cut of size smaller than δ in G. Then every dominating set U of G has vertices in V_1 and in V_2*

Proof: Suppose this is false and $U \subseteq V$. Choose arbitrarily, a vertex $v \in V_2$ and let $v_1, v_2, ..., v_\delta$ be δ of its neighbors. For each $i \in \{1, ..., \delta\}$, define an edge e_i of the given cut as follows. If $v_i \in V_1$, then $e_i = \{v, v_i\}$, otherwise $v_i \in V_2$, and since U is dominating, there is at least on vertex $u \in U$ such that $\{u, v_1\}$ is an edge. Take this u and put $e_i = \{u, v_i\}$. The δ edges $e_1, ..., e_\delta$ are all distinct and lie in the given cut, contradicting the assumption that its size is less than δ. ■

1.3 Combinatorics

Definition 1.9. *A hypergraph is a pair $H = (V, E)$, where V is a finite set whose elements are called vertices, and E is a family of subsets of V called edges. It is n-uniform if each of its edges contains precisely n vertices. We say that H has property B, or that it is two-colorable, if there is a two-coloring of V such that no edge is monochromatic. Let $m(n)$ denote the minimum possible number of edges of an n-uniform hypergraph that does not have property B.*

Proposition 1.10. *Every n-uniform hypergraph with less than 2^{n-1} edges has property B. Therefore $m(n) \geq 2^{1-n}$*

Proof: Let $H = (V, E)$ be an n-uniform hypergraph with less than 2^{n-1} edges. Color V randomly two colors. For each edge $e \in E$, let A_e be the event such that e is monochromatic. Clearly, $Pr[A_e] = 2^{1-n}$. This is because there are two monochromatic colors our edge can take and this edge connects to n vertices. Therefore $Pr[\bigcup_{e \in E} A_e] \leq \sum_{e \in E} Pr[A_e] < 1$ by Union Bound Inequality. Since our probability is less than 1, the event exists and thus there is a two-coloring without monochromatic edges. ■

Let us now try to find the best known upper bound for m(n). Let us fix V and pick v vertices in V. Denote X to be a coloring of V which has "a" points in one color and then "$b = v - a$" points in a second color. Let us then define the set S to be a uniformly random selection n points from V. We can then find the probability that S is monochromatic under our coloring X. We can see that we have $\binom{v}{n}$ total possibilities of vertex colors for our set S as we are essentially choosing n points to color out of v total points and we want every possible combination of these n points. We can see the portion of our "n" points that are strictly colored in our "a" color is $\binom{a}{n}$ because we want all possible combinations of choosing n vertices and they're all in the "a" color. Similarly we can define $\binom{b}{n}$ to be our second coloring. We can add these two together to get our true probability. Thus Pr[S is monochromatic in X] $= \frac{\binom{a}{n}+\binom{b}{n}}{\binom{v}{n}}$. For our sake of argument we assume that v is an even amount of vertices. $\binom{y}{n}$ is convex which we will use without proof. However, since it is convex, we can see our probability is minimized at $a = b$. This is easy to see due to convexity. Thus we can find a lower bound for our probability to be $p = \frac{2*\binom{v/2}{n}}{\binom{v}{n}}$. We next define S_i to be uniformly and independently determined sets just as we have for S. We do this for some "m" such that we have $S_1, .., S_m$ different sets. We now define A_X to be the event that in which all of our sets are not monochromatic. We can see that $\Pr[A_X] \leq (1-p)^m$. This is because a single A_X is monochromatic with at least probability p so the probability that none of them are monochromatic is simply $(1-p)$. Since we have done this "m" types, the probability is appropriately $(1-p)^m$. We have "v" total vertices, each of which can take on a different color. Thus there are a total of 2^v different colors. We can see that Pr[V A_X] $\leq 2^v(1-p)^m$. This is an upper bound for this union because at most 2^v of these coloring's will be used in the unions of our A_X

Theorem 1.11. $m(n) < (1+o(1))\ \frac{eln(2)}{4} n^2 2^n$

Definition 1.12. *Let $F = \{(A_i, B_i)\}_{i=1}^h$ be a family of pairs of subsets of an arbitrary set. We say that F is a (k, l) system if $|A_i| = k, |B_i| = l$ for all $1 \leq i \leq h$, $A_i \cap B_i = \emptyset$, $A_i \cap B_j \neq \emptyset$*

Theorem 1.13. *If F defined above is a (k-l) system, then $h \leq \binom{k+l}{k}$*

Proof: We first define $X = \bigcup_{i=1}^h (A_i \cup B_i)$ and take a random ordering of X denoted as π. For every union of A_i, B_i in our big union, we let X_i to be the event that every element of A_i precede every element in B_i. $\Pr[X_i] = 1/\binom{k+l}{k}$. This number comes because we have a total of $k+l$ numbers. This is given to us as this is a (k,l) system which means A_i, B_i have k and l elements respectively. We are also choosing k elements from this set because we are choosing k elements and then filling the rest of the elements arbitrarily. Since only one specific ordering has the order we want, it is easy to see our probability is correct. We can also prove that X_i, X_j is pairwise disjoint with contradiction. Assume that they are not and that this is false. If the last element of A_i comes before the last element of A_j, we see that A_i precedes all of B_j which is a contradiction as by definition, a (k,l) system requires $A_i \cap B_j \neq \emptyset$. We see that Pr[V$X_i$] $= \sum_{i=1}^h Pr[X_i] = h/\binom{k+l}{k}$. This completes the proof as this is probability is less than 1 and therefore we can say that $h \leq \binom{k+1}{k}$ ■

2 Exercises

2.1 9/27/24 Exercise

Consider the 3SAT problem. For any set of clauses, there exists an assignment that satisfies at least 7/8 of the clauses.

Proof: Let us take a random sample of clauses. We know the probability of a clause being unsatisfiable is 1/8 as we have 7 combinations of clauses that can satisfy given 3 variables and their negations. Since this probability is nonzero, we have shown there exists an assignment that satisfies at least 7/8 of the clauses.

2.2 Chapter 1 Exercise 1

2.3 Chapter 1 Exercise 7

Theorem 2.1. *Let $\{(A_i, B_i), 1 \le i \le h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i and $|B_i| = \ell$ for all i, $A_i \cap B_i = \emptyset$, and $(A_i \cap B_j) \cup (A_j \cap B_i) \ne \emptyset$ for all $i \ne j$. Prove that $h \le (k+\ell)^{k+\ell}/(k^k \ell^\ell)$.*

Proof. We will randomly color the integers using $k+\ell$ colors. Formally, we say that integer x has color $\pi(x)$, where $\pi(x)$ is chosen uniformly at random from the set $\{1, \dots, k+\ell\}$.

For each $i, 1 \le i \le h$, let X_i be the event that all the elements of A_i have colors from $\{1, \dots, k\}$ and all the elements of B_i have colors from $\{k+1, \dots, \ell\}$.

Lemma 2.2. *The probability that event X_i happens is*

$$\Pr[X_i] = \frac{k^k \ell^\ell}{(k+\ell)^{k+\ell}}.$$

Proof. Let us create k + l bins, representing the elements in $A_i \cup B_i$. We can see that there are $(k+l)^{k+l}$ different ways to place a color in a bucket. However, we also know that there are k^k ways to place the k colors in A to the first k buckets and l^l ways to place the l colors in B to the last l buckets. Thus we can see the probability that event X_i happens is $\frac{k^k l^l}{(k+l)^{k+l}}$ □

Lemma 2.3. *The events X_i are disjoint.*

Proof. Let us proceed by contradiction. If X_i, X_j were not disjoint, it means that A_i, A_j are colored from the same colors and same with B_i, B_j which is a contradiction as the coloring of the elements must be done drawn from a set so we cannot color two sets at the same time as we run out of colors. □

It follows that

$$1 \ge \Pr[\vee_{i=1}^h X_i] = \sum_{i=1}^h \Pr[X_i] = h \cdot \frac{k^k \ell^\ell}{(k+\ell)^{k+\ell}}.$$

Thus

$$h \le \frac{(k+\ell)^{k+\ell}}{(k^k \ell^\ell)},$$

completing the proof.

□

3 Linearity of Expectation

3.1 Basics

Definition 3.1. *Given random variables,* $X_1, ..., X_n$ *with* $X = c_1X_1 + ... + c_nX_n$*. The Linearity of Expectation is* $E[X] = c_1E[X_1] + ... + c_nE[X_n]$

Definition 3.2. *A Hamiltonian Path is one that visits every vertex exactly once.*

Theorem 3.3. *There is a tournament T with n players and at least* $n!2^{-(n-1)}$ *Hamiltonian paths.*

Proof. Let us create a random tournament and denote X to be the number of Hamiltonian paths of the graph. Denote σ and X_σ to be the indicator random variable that there is a Hamiltonian path for a given permutation. $X = \sum X_\sigma$. There are $n!$ different permutations that give a valid Hamiltonian path and there are 2^{n-1} different permutations. Thus $E[X] = n!/2^{n-1}$ □

3.2 Splitting Graphs

Theorem 3.4. *Let* $G = (V, E)$*, be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least* $e/2$ *edges*

Proof. Let T be a random subset of V given by $Pr[x \in T] = 1/2$. Let $B = V - T$. Denote an edge to be crossing if $e = \{x, y\}$ and exactly one of x or y is in T. We then denote X to be the number of crossing edges. $X = \sum X_{xy}$ where X_{xy} is an indicator variable denoting that an edge between x and y is crossing. Thus $E[X_{xy}] = 1/2$. This is because there is a 1/2 chance that a given edge is crossing. Thus we can see that $E[X] = \sum E[X_{xy}] = e/2$ □

Theorem 3.5. *If G has 2n vertices and e edges, then it contains a bipartite subgraph with at least* $en/(2n-1)$ *edges. If G has* $2n+1$ *vertices and e edges, then it contains a bipartite subgraph with at least* $e(n+1)/(2n+1)$ *edges*

Proof. When G has 2n vertices, we can let T be uniformly chosen from all n size subsets of V. Thus every edge now has a probability $n/(2n-1)$ of being crossing. We can do the same thing for if G has 2n + 1 vertices. □

Lemma 3.6. *Let* P_k *denote the set of all homogeneous polynomials of degree k with all coefficients having absolute value at most 1. For all* $f \in P_k$ *there exist* $p_1, .., p_k \in [0, 1]$ *with* $|f(p_1, ..., p_k)| \geq c_k$

Proof. We set $M(f) = max|f(p_1, ..., p_k)|$. For every $p_k \in P_k$, $M(f) > 0$ as f is not the zero polynomial. Since P_k is compact we can see that $M : P_k \to R$ is continuous and c_k must be the minimum. □

3.3 Two Quickies

Theorem 3.7. *There is a two-coloring of* K_n *with at most* $\binom{n}{a}2^{1-\binom{n}{a}}$ *monochromatic* K_a

Proof. Let us create a random coloring of K_n. Define X to be the number of monochromatic K_a and find $E[X]$. For some coloring, the value of X is at most this expectation. We can see that $E[X] = \binom{n}{a}2^{1-\binom{n}{a}}$. This is because there are $\binom{n}{a}$ different K_a and then each K_a has an expectation of $2^{1-\binom{n}{a}}$. □

Theorem 3.8. *There is a two-coloring of $K_{m,n}$ with at most $\binom{m}{a}\binom{n}{b}2^{1-ab}$ monochromatic $K_{a,b}$*

Proof. Take a random coloring of $K_{m,n}$ and define X to be the number of monochromatic $K_{a,b}$ and find $E[X]$. $E[X] = \binom{m}{a}\binom{n}{b}2^{1-ab}$. Similar reasoning to the above proof. □

3.4 Balancing Vectors

Theorem 3.9. *Let $v_1, ..., v_n \in R^n$, all $|v_i| = 1$. Then there exist $e_1, ..., e_n \in -1, 1$ so that $|e_1v_1 + ... + e_nv_n| \leq \sqrt{n}$ and also there exist $e_1, ..., e_n \in 1, 1$ so that $|e_1v_1 + ... + e_nv_n| \geq \sqrt{n}$*

Proof. Let us select $e_1, ..., e_n$ uniformly and independently from $-1, 1$. Set $X = |e_1v_1 + ... + e_nv_n|^2$. Then we can see that $X = \sum_{i=1}^{n}\sum_{j=1}^{n} e_ie_jv_iv_j$. We can therefore see that $E[X] = \sum_{i=1}^{n}\sum j = 1^n v_iv_jE[e_ie_j]$. We can therefore simplify this to $E[X] = \sum_{i=1}^{n} v_iv_i = n$. Thus we can take the square root of X to prove the theorem. □

Theorem 3.10. *Let $v_1, ..., v_n \in R^n$, all $|v_i| \leq 1$. Let $p_1, ..., p_n \in [0, 1]$ be arbitrary, and set $w = p_1v_1 + ... + p_nv_n$. Then there exist $e_1, ..., e_n \in 0, 1$, so that setting $v = e_1v_1 + ... + e_nv_n$. $|w - v| \leq \frac{\sqrt{n}}{2}$*

Proof. Let us pick the e_i independently with $Pr[e_i = 1] = p_i$ and $Pr[e_i = 0] = 1 - p_i$. Define random variable $X = |w - v|^2$. Therefore we can see that $X = \sum_{i=1}^{n}\sum_{j=1}^{n} v_iv_j(p_i - e_i)(p_j - e_j)$. Thus the $E[X] = \sum_{i=1}^{n}\sum_{j=1}^{n} v_iv_jE[(p_i - e_i)(p_j - e_j)]$. If $i \neq j$, our expectation is 0. If $i = j$, then our expectation is equal to $p_i(p_i - 1)^2 + (1 - p_i)p_i^2 \leq \frac{1}{4}$. We know that $E[(p_i - e_i)^2] = Var[e_i]$. This shows the expected value of X is less than or equal to $\frac{n}{4}$ □

4 Alterations

4.1 Combinatorial Geometry

Theorem 4.1. *There is a set S of n points in the unit square U such that $T(S) \geq 1/(100n^2)$*

4.2 Packing

Theorem 4.2. *Let C be bounded, convex, and centrally symmetric around the origin. Then $\delta(C) \geq 2^{-d-1}$*

Proof. Let P,Q be independently and randomly sampled from $B(x)$ where $B(x)$ is the cube $[0,x]^d$ of side x. Consider the event $(C+P) \cap (C+Q) \neq \emptyset$. This can only happen if there is some $c_1, c_2 \in C$ such that $P - Q = c_1 - c_2 = 2\frac{c_1 - c_2}{2} \in 2C$ by central symmetry and convexity. CONTINUE □

4.3 Greedy Coloring

Corollary 4.3. $m(n) = \Omega(2^n (n/\ln n)^{1/2})$

Proof. Let us bound $1 - p \leq e^{-p}$. The function $ke^{-pn} + k^2 p$ is minimized at $p = \ln(n/k)/n$. Thus we have $k^2/n(1 + \ln(n/k)) < 1$. □

Theorem 4.4. *If there exists $p \in [0,1]$ with $k(1-p)^n + k^2 p < 1$ then $m(n) > 2^{n-1}k$*

5 The Second Moment

5.1 Basics

Definition 5.1. *We define Variance for a random variable X to be $Var[X] = E[(X - E[X])^2]$*

Theorem 5.2. *Chebyshev's Inequality states that for any positive λ, $Pr[|X - \mu| \geq \lambda\sigma] \leq \frac{1}{\lambda^2}$*

Proof. $\sigma^2 = Var[X] = E[(x - E[X])^2 \geq \lambda^2\sigma^2 Pr[|X - \mu| \geq \lambda\sigma]$ □

Chebyshev's essentially bounds how many data points are λ standard deviations away from the mean.

Definition 5.3. *We define Covariance for two random variables X and Y to be $Cov[X, Y] = E[XY] - E[X]E[Y]$*

5.2 Number Theory

Theorem 5.4. $|v(x) - \ln\ln n| > \omega(n)\sqrt{\ln\ln n}$

Proof. Denote x to be randomly chosen from a set $\{1, ..., n\}$. Denote X_p to be 1 if $p|x$ and 0 otherwise. Set $M = n^{1/10}$ and have $X = \sum X_p$. $E[X_p] = \frac{n/p}{p} = 1/p + O(1/n)$. Thus we can see that $E[X] = \ln\ln n + O(1)$ using the fact that $\sum 1/p = \ln\ln n$. We know that $Var[X] = \sum Var[X_p] + \sum Cov[X_p, X_q]$. We know $Var[X_p] = (1/p)(1 - 1/p)$ so therefore $Var[X_p] = \ln\ln n + O(1)$. $Cov[X_p, X_q] = E[X_p, X_q] - E[X_p]E[X_q] = \frac{n/pq}{n} - \frac{n/p}{n}\frac{n/q}{n} \leq \frac{1}{n}(\frac{1}{p} + \frac{1}{q})$. Thus $\sum Cov[X_p, X_q] \leq \frac{2M}{n}\sum\frac{1}{p} \leq O(n^{-9/10}\ln\ln n)$. Thus by Chebyshev's we have that $Pr[|X - \ln\ln n| > \lambda\sqrt{\ln\ln n}] < \lambda^{-2} + O(n^{-9/10}\ln\ln n)$. □

5.3 More Basics

Theorem 5.5. $Pr[X = 0] \leq \frac{Var[X]}{E[X]^2}$

Proof. Set $\lambda = \mu/\sigma$. We can then use Chebyshev's Inequality to see that $Pr[X = 0] \leq Pr[|X - \mu| \geq \lambda\sigma] \leq \sigma^2/\mu^2$ □