DRP: 321-Avoiding Permutations

Jasmine Latimer

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1 Overview

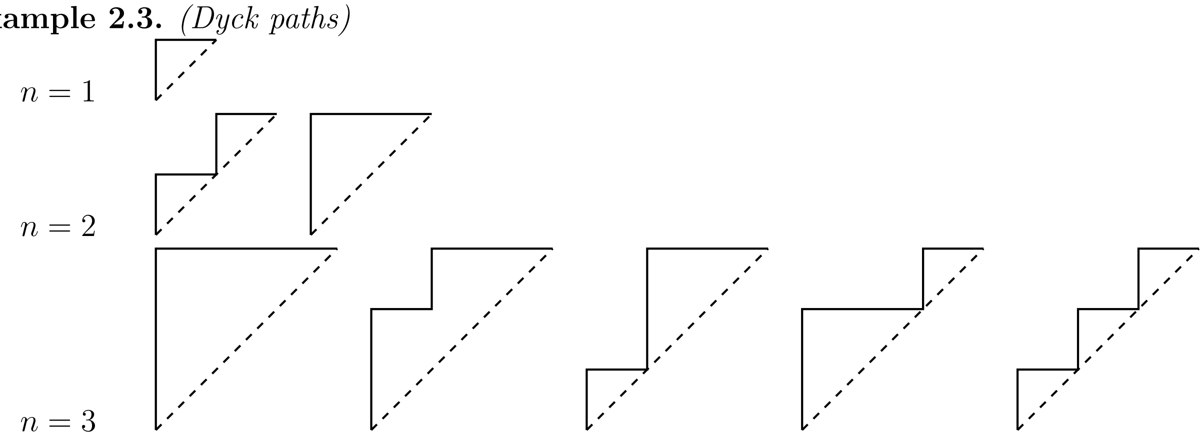
This semester, I learned a lot about combinatorics with my mentor Mitsuki and Emma. By reading Bruce E. Sagan's *Combinatorics: The Art of Counting*, we covered fun topics like sign-reversing involutions, trees, and pattern avoidance. I decided to focus my final paper on 321-avoiding permutations, providing two different bijections between 321-avoiding permutations and Dyck paths, one of the most famous Catalan objects.

2 Catalan Numbers

Definition 2.1. **Catalan numbers** are defined recursively by $C_0 = 1$, $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$.

Definition 2.2. A **Dyck path** of length 2n is a lattice path from (0, 0) to (n, n) that stays above or on the diagonal.

Example 2.3. *(Dyck paths)*



Theorem 2.4. *The number of Dyck paths of length 2n is counted by the Catalan number C_n.*

Proof. (by induction)

For the base case, there is only one way to make a Dyck path of length 0. Thus, $C_0 = 1$.

For the recursive case, suppose the number of Dyck paths of length 2n is counted by the Catalan number C_n. Let T be a Dyck path of length 2(n + 1) and let (2k, 2k) be the first point where T touches the diagonal and $k \neq 0$. The segment of T from (0, 0) to (2k, 2k) is a Dyck path of length 2(k - 1) that starts with an up step and ends with a right step. The segment from (2k, 2k) to (2(n + 1), 2(n + 1)) is a Dyck path of length 2(n + 1 - k) Then, the number of Dyck paths of length 2(k - 1) is counted by C_{k-1} and the number of Dyck paths of length 2(n + 1 - k) is counted by C_{n+k-1}. Since k ranges from 1 to n + 1 and by shifting the index,

$$C_{n+1} = \sum_{k=1}^{n+1} C_{k-1}C_{n+1-k} = \sum_{k=0}^{n} C_k C_{n-k} \tag{2.1}$$

□

Definition 2.5. A **321-avoiding permutation** is a permutation $a_1a_2 \ldots a_n$ with its longest decreasing sequence of length less than three (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$).

Example 2.6. *(321-avoiding permutations)*

$n = 1$	1
$n = 2$	12, 21
$n = 3$	123, 213, 132, 312, 231
$n = 4$	1234, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2341, 2413, 3124, 3142, 3412, 4123

3 Bijection Using Left-to-Right Maxima

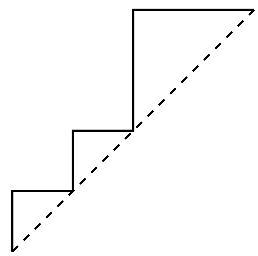
Claim 3.1. *There exists a bijection between 321-avoiding permutations and Dyck paths.*

Proof. Given a permutation σ, we first distinguish left-to-right maxima. Then, we can divide the permutation into subsets that start with a maxima and are followed by non-maxima. For each subset, the relative magnitude of the maxima compared to all the elements to the right determines how many steps up we take in the Dyck path. For instance, if the maxima is the second smallest out of all the elements to the right, we take as many steps up until we are two units above the diagonal in the Dyck path. The size of each subset corresponds to how many steps to the right we take. In 321-avoiding permutations, the non-maxima occur in increasing order, so the Dyck path is uniquely determines by the 321-avoiding permutation. □

Example 3.2. *What Dyck path does the 321-permutation* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ *correspond to?*

The left-to-right maxima of σ = 1243 are 1, 2, and 4. Then, the permutation can be split into (1)(2)(43). Since 1 is the smallest element out of all of the elements to the right, we take one step up in the Dyck path followed by one step to the right since 1 is in a group alone. We also take one step up and one step to the right for 2. Since 4 is the second smallest out of the elements to the right, we take steps up until we are two units above the diagonal. There are two elements in the group, so we take two steps to the right and end at (4, 4). Two important notes are that the rank of the left-to-right maxima correspond to distance above the diagonal instead of the number of steps up, and we determine rank out of the remaining elements to the right.

Thus, the 321-permutation σ corresponds to the Dyck path below that first takes two one-unit steps and then one two-unit step.



4 Bijection Using RSK Algorithm

Definition 4.1. Robinson-Schensted-Knuth (RSK) Algorithm takes in a permutation and outputs two standard Young tableaux (SYT).

Given a permutation $\sigma \in S_n$ where $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$, start with two empty SYTs, P_0 and Q_0. For P, insert σ(i), $1 \leq i \leq n$, into P_{i-1} with the following process:

- Look for an entry that has a greater value than σ(i) in the first row. If no entry is found, append σ(i) to that row.
- If such an entry is found, replace the entry with σ(i) and repeat the process with the original entry in the next row.
- If the row is empty, create a new row with σ(i).

For Q, insert i into Q_{i-1} to get Q in the same position as the last position where an entry was placed in P_{i-1} to get P_i.

Theorem 4.2. *The shape of the SYT encodes information about the permutation, i.e. the number of boxes in each row corresponds to the length of increasing subsequences, and the number of boxes in each column corresponds to the length of decreasing subsequences.*

Claim 4.3. *(alternative to 3.1) There exists a bijection between 321-avoiding permutations and Dyck paths.*

Proof. Given a 321-avoiding permutation $\sigma \in S_n$, we can apply the RSK algorithm to the permutation to obtain two SYTs, P and Q, of the same shape. Then, we can replace each number k in Q by 2n + 1 - k, denoted by Q'.

By Theorem 4.2, since the permutation is 321-avoiding, then the length of the longest column is at most 2. Then, the shape of the SYT in the pair is either one row of four boxes, two rows of two boxes, or two rows with the top having three boxes and the bottom having one. Since the two SYTs always have the same shape, we can always form a two-by-four SYT denoted R by rotating Q' 180 degrees and attaching it to P.

Thus, we can then obtain a Dyck path from R by taking a step up if i is in the first row and to the right if i is in the second row for i = 1, ..., 2n. The path will not cross the diagonal because R is strictly increasing both left-to-right and up-to-down. □

Example 4.4. *(alternative method to example 3.2) What Dyck path does the 321-permutation* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ *correspond to?*
First, we can apply the RSK algorithm to σ.

P_1

2	

Q_1

1	

P_2

2	4

Q_2

1	2

P_3

1	4
2	

Q_3

1	2
3	

P_4

1	3
2	4

Q_4

1	2
3	4

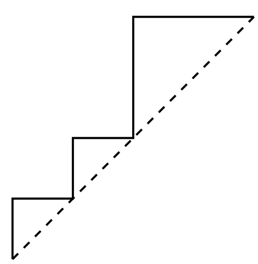
Then, we can replace each number in Q by 2n + 1 - k.

8	7
6	5

By rotating the new SYT by 180 degrees and attaching it to P, we get a new SYT that has two rows and four boxes in each row.

1	3	5	6
2	4	7	8

We obtain a Dyck path from the new SYT by taking a step up if k is in the first row and to the right if k is in the second for for k = 1, . . ., 8.



One result that we find is that both bijections result in the same Dyck path for the permutation (1243). We would have to prove this further, but it turns out that this is true for all 321-avoiding permutations. Also, a natural question that comes up is whether we can extend the RSK algorithm to other avoidance permutations like 312-avoiding permutations. We could only do so if the pair of SYTs have the correct shape so that they can be connected to form a two-by-2n SYT. Thus, a bijection using the RSK algorithm may not be used for all types of avoidance permutations.

References

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