Stochastic Differential Equations Summary

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1 Brownian Motion

1.1 Motivation

Brownian motion allows us to mathematically justify the behavior of randomized processes. By defining a mathematical function that makes use of normal distributions to represent some quantitative measure of an object, we are better suited to model and represent quantities which include "randomized" measures. Our eventual goal comes back to a general differential equation as shown in Oksendal's *Stochastic Differential Equations*, understanding an equation of the form

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“noise”}$$

Here, the differential equation is altered in the fact that it has some randomized "noise" being added to the otherwise regular differential equation.

1.2 Definition

We define Brownian Motion as is explained in Lawrence C. Evans *An Introduction to Stochastic Differential Equations*. A function $B(t)$ is defined as Brownian Motion if:

1. $B(0) = 0$. Essentially, the function should begin with no randomization.

2. $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t - s$ given $t \geq s \geq 0$. There should be no expectation for the function to predictably change.

3. For any partition of times $(t_1, t_2, \cdots, t_n)$, the random variables $B(t_1), B(t_2) - B(t_1), B(t_3) - B(t_2), \cdots, B(t_n) - B(t_{n-1})$ should all be independent of each-other. To encourage a fully randomized process, the increments of the function should not be linked to each-other.

Proving that such a construction exists is necessary, but is not in the scope of this review.

2 Constructing the Ito Integral

2.1 Motivation

The eventual goal is to make sense of the aforementioned differential equation

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“noise”}$$

We might want to write this “noise” term as being represented by some stochastic process S_t, giving us

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot S_t$$

We can treat the differential equation as being made up of discrete sums

$$X_k - X_{k-1} = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)S_k\Delta t_k$$

This $S_k\Delta t_k$ term can be represented by some ΔV_k, where V_k has stationary independent increments at mean 0. Using Brownian motion as a sufficient replacement for V_k, we then replace $S_k\Delta t_k$ with ΔB_k. We then get that

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_k$$

Which in integral form becomes

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

Now, the motivation of conducting the Ito integral comes in making sense of this $\int_0^t \sigma(s, X_s)\, dB_s$ term.

2.2 What is the Ito Integral?

We generally find that this term has behaviors that do not necessarily make sense given the classic definition of an integral. We take two different approximations of $B_t(\omega)$. We have

$$\phi_1(t, \omega) = \sum_{j \geq 0} B_{j \cdot 2^{-n}} \cdot \chi_{[j \cdot 2^{-n}, (j+1) \cdot 2^{-n})}(t)$$

$$\phi_2(t, \omega) = \sum_{j \geq 0} B_{(j+1) \cdot 2^{-n}} \cdot \chi_{[j \cdot 2^{-n}, (j+1) \cdot 2^{-n})}(t)$$

for χ being the characteristic function and n being some natural number. Essentially, we have left and right side approximations of some Brownian motion function B_j. Integrating over both of these gives

$$E[\int_0^T \phi_1(t,\omega)dB_t(\omega)] = \sum_{j\geq 0} E[B_j(B_{j+1}-B_j)] = 0$$

The expected value comes out to 0 since we know Brownian motion has independent increments. However, for $\phi_2(t,\omega)$, we have

$$E[\int_0^T \phi_2(t,\omega)dB_t(\omega)] = \sum_{j\geq 0} E[B_{j+1}(B_{j+1}-B_j)] = \sum_{j\geq 0} E[(B_{j+1}-B_j)^2]+\sum_{j\geq 0} E[B_j(B_{j+1}-B_j)]$$

$$= \sum_{j\geq 0} E[(B_{j+1}-B_j)^2] = T$$

The proof for this is as follows:

We know $B_{j+1}-B_j$ has variance $t_{j+1}-t_j = E((B_{j+1}-B_j)^2)-(E(B_{j+1}-B_j))^2 = E((B_{j+1}-B_j)^2)$. Thus, we know $\sum_{j\geq 0} E[(B_{j+1}-B_j)^2] = \sum_{j\geq 0} t_{j+1}-t_j = T$.

This means that we need to determine some convention to confirm that these integrals are well-defined. We decide to use the left-hand endpoint, which we define as the Ito Integral.

2.3 Justification

The decision behind choosing the left endpoint is because its construction can be classified as a martingale. A martingale X_t is defined as follows:

1. $E[|X_t|] < \infty$ for all $t \geq 0$.
2. $E[X_t|\mathcal{U}_s] = X_s$ for $t \geq s \geq 0$.

   In this case, $\mathcal{U}_s$ is the σ algebra generated by all $X(a)$ for $0 \leq a \leq s$, essentially the history of the process up to s. Given a left-hand endpoint, we can be confident that $E(\int_0^t \sigma(s,X_s)\,dB_s|\mathcal{U}_r) = \int_0^r \sigma(s,X_s)\,dB_s$ for $t \geq r \geq 0$, since each added point has expected value 0 being multiplied by $(B_{j+1}-B_j)$.

The equation in higher dimensions works the exact, same, with

$$B(t,\omega) = \begin{pmatrix} B_1(t,\omega) \\ \vdots \\ B_n(t,\omega) \end{pmatrix}, \; u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$
$$v = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix}, \; dB(t) = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{pmatrix}$$

We then have that $dX(t) = udt + vdB(t)$.

3 Existence

To prove that we have convergence for such a left-hand approximation, we show that

$$\int_0^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_0^T \phi_n(t,\omega)dB_t(\omega)$$

given that

$$E[\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2 dB_t(\omega)] \to 0$$

These $\phi_n(t,\omega)$ are elementary, meaning they are representable as

$$\sum_j e_j(\omega) \cdot \chi_{[t_j,t_{j+1})}(t)$$

We can then use these functions to approximate our original function, and we can state the Ito Isometry for them, which is that

$$E[(\int_0^T \phi(t,\omega)dB_t(\omega))^2] = E[\int_0^T \phi(t,\omega)^2 dt]$$

The proof is because $E[(\int_0^T \phi(t,\omega)dB_t(\omega))^2] = \sum_{i,j} E[e_i e_j \Delta B_i \Delta B_j] = E[e_j^2 \cdot (t_{j+1} - t_j)] = E[\int_0^T \phi^2 dt]$

We can then use this isometry to state that if we have functions ϕ_n such that

$$E[\int_0^T (f - \phi_n)^2 dt] \to 0$$

then

$$\int_0^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_0^T \phi_n(t,\omega)dB_t(\omega)$$

This is convergent, since $\int_0^T \phi_n(t,\omega)dB_t(\omega)$ forms a Cauchy sequence, shown by the Ito Isometry. From here, we can then even extend the Iso Isometry to non-elementary functions.

4 Ito Representation Theorem

We know that the Ito Integral acts as a sufficient martingale with respect to the σ algebra generated to represent the history of the object up to time t. The goal of this section is to prove that any square integrable has a representation as an Ito integral. To put it in mathematically formal terms, the Ito Representation Theorem states that if some function $F \in L^2(\mathcal{F}_T^{(n)}, P)$ being the set of square integrable functions with history $\mathcal{F}_T^{(n)}$, then it has representation as $F(\omega) = E[F] + \int_0^T f(t,\omega)\, dB(t)$ for stochastic process $f(t,\omega)$. The proof is as follows:

We want to show that any $F \in L^2(\mathcal{F}_T^{(n)}, P)$ can be represented by a linear combination of functions of the sort $\exp\{\int_0^T h(t)dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)dt\}$, where $h(t) \in L^2[0,T]$. Without going into full detail, Oksendal's paper explains why these sorts of functions are expressable in terms of an Ito integral.

Now take some function F_n, expressed as a linear combination of functions of the form $\exp\{\int_0^T h(t)dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)dt\}$. We then know that

$$F_n(\omega) = E[F_n] + \int_0^T f_n(s,\omega)dB_s(\omega)$$

The Ito Isometry states that for elementary function $\phi(t,\omega)$ bounded, we have

$$E[(\int_0^T \phi(t,\omega)dB_t(\omega))^2] = E[\int_0^T \phi(t,\omega)^2 dt]$$

This implies that

$$E[(F_n - F_m)^2] = E[(E[F_n - F_m] + \int_0^T (f_n - f_m)dB)^2] = E[F_n - F_m]^2 + \int_0^T E[(f_n - f_m)^2]dB$$

However, as $n, m \to \infty$, we know that $(f_n - f_m)^2$ will have expected value 0, meaning that we have that f_n is a Cauchy sequence and thus converges to some f. Thus, we have

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty}(E[F_n] + \int_0^T f_n dB) = E[F] + \int_0^T f dB$$

Thus, we have a representation for function F. This representation will actually be unique, but the proof will not be shown here.