# REPRESENTATIONS OF THE LIE ALGEBRAS $\mathfrak{sl}_n(\mathbb{C})$

AREN MARTINIAN

Abstract. In this survey paper we state and prove results about finite dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$ and briefly discuss Schur-Weyl Duality and representation theory of the symmetric group $\mathcal{S}_n$. In particular, every finite-dimensional representation of $\mathfrak{sl}_n(\mathbb{C})$ is a direct sum of irreducible, or simple representations $L(\lambda)$ parameterized by $\lambda \in \Lambda^{dom}$, and furthermore $L(\lambda)$ contains a vector of "highest weight" $v_\lambda$.

## 1. Introduction

The Mathematics Directed Reading Program (or DRP) pairs a graduate student in the department with an undergraduate to assist in learning a particular topic. In our case, the topic was chosen to be finite dimensional semi-simple representations of Lie Algebras (specifically $\mathfrak{sl}_n(\mathbb{C})$), with a brief digression on Schur-Weyl duality and representations of the symmetric group $\mathcal{S}_n$.

Most of the results in this paper will follow from [3], and material will be presented in a format very similar to how it is done in these lectures. An attempt will be made to summarize the DRP meetings chronologically, with exception to the conversation on Schur-Weyl duality, which will be relegated to the end of the paper. Additional materials used are [1, 2].

This material is being presented in a paper format as opposed to a presentation due to the universality of the material taught. The intent of writing a paper in such a way is to synthesize the material presented in a manner that is accessible and useful to look back upon if needed.

## 2. Preliminaries

Our goal is to study finite-dimensional representations of Lie Algebras. In order to do this, we introduce the following definitions for the following discussion and to motivate how one obtains Lie Algebras in general.

**Definition 2.1.** A *Lie Group* is a group $G$ that is also a differentiable manifold. (In particular, we can consider its tangent space at the identity $T_I(G)$)

Given a Lie Group $G$, a natural question to ask is: what structure does $\mathfrak{g} = T_I(G)$ have?

**Definition 2.2.** A *Lie Algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie Bracket* satisfying the following properties:

(1) $[-,-]$ is antisymmetric: $\forall x, y \in \mathfrak{g},\ [x, y] = -[y, x]$
(2) $[-,-]$ satisfies the Jacobi identity: $\forall x, y, z \in \mathfrak{g},\ [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Without delving into the rather tedious checks in general, it turns out that $\mathfrak{g} = T_I(G)$ is a Lie Algebra. Instead, we give the following example:

**Example 2.3.** Let $G = SL_2(\mathbb{C})$, i.e. the special linear group ($2 \times 2$ matrices of determinant 1) over the complex numbers. We claim that $T_I(SL_2(\mathbb{C}))$ is a Lie Algebra:

*Date*: May 14, 2023.

*Key words and phrases.* Lie Groups, Lie Algebras, Representations.

*Proof.* Consider any matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then for some small $\epsilon > 0$, we can take $I + \epsilon X$ and consider what restrictions must be placed on $X$ in order for the determinant to still be 1.

But $I + \epsilon X = \begin{bmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon d \end{bmatrix}$

And $\det\{(I + \epsilon X)\} = 1 + \epsilon a + \epsilon d - \epsilon^2(ad - bc) = 1 + \mathrm{tr}(X) + O(\epsilon^2)$.

Thus we see that in order for $\det\{(I + \epsilon X)\}$ to be 1 to first order, we must have $\mathrm{tr}(X) = 0$. In other words, $T_I(SL_2(\mathbb{C})) = \{\text{2x2 matrices}\, X \text{with trace } 0\} = \mathfrak{sl}_2(\mathbb{C})$. It is easy to check that this is a vector space, and that $[X, Y] = XY - YX$ defines a Lie Bracket. ☐

It turns out that $T_I(GL_2(\mathbb{C}))$ is $\mathrm{Mat}(2 \times 2)$, the vector space of all $2 \times 2$ matrices. We will not be focusing on this Lie Algebra, however, as $\mathfrak{sl}_n(\mathbb{C})$ is more interesting.

We are now ready for our next definition:

**Definition 2.4.** Given a (finite dimensional) vector space $V$, a *representation* of a Lie Algebra $\mathfrak{g}$ is a linear map

$$\mathfrak{g} \to End(V)$$

Where $End(V)$ denotes the endomorphism ring of $V$, such that $\forall X, Y \in \mathfrak{g}, \forall v \in V$,

$$[X, Y]v = XYv - YXv$$

**Example 2.5.** Let $\mathfrak{g} = \mathbb{C}$. Then $[1, 1] = -[1, 1] = 0$. Then a representation $\mathbb{C} \to End(V)$ is uniquely determined by linearity where it sends 1:

Since $\mathbb{C}$ is abelian, for any $a, b \in \mathbb{C}$, $[a, b]x = abx - bax = 0$. This tells us that the representations coincide with all endomorphisms of V.

## 3. Classifying Finite-Dimensional Representations of $\mathfrak{sl}_2$

First, we will denote $\mathfrak{sl}_2(\mathbb{C})$ simply by $\mathfrak{sl}_2$, where the complex numbers is implied.

Let us begin by stating the main theorem for this section, and we will build up toward the proof.

**Theorem 3.1.** *Every finite-dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations (i.e. has no nontrivial submodule) denoted $L(n)$.*

Note that the property that every finite-dimensional representation is a direct sum of irreducibles is called *semisimplicity.*

Before we begin with a proof, here is a very important nonexample:

**Non-Example 3.2.** The Lie Algebra $\mathfrak{g} = \mathbb{C}$ has a nontrivial 2-dimensional representation sending 1 to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in End(\mathbb{C}^2)$ This representation is not semisimple! $\mathbb{C}e_1$ is a subrepresentation, but $\mathbb{C}e_2$ is certainly not, since it is not closed under the operations of $\mathfrak{g}$.

To gain familiarity toward our general proof, we will choose a particular presentation as follows: Define the following elements of $\mathfrak{sl}_2$:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is not hard to check that the following relations hold:

- $[e, f] = h$
- $[h, e] = 2e$
- $[h, f] = -2f$

More generally, *every* finite-dimensional representation of $\mathfrak{sl}_2$ has the elements $e, f, h$ satisfying the same relations, but they will have different presentations.

Additionally, we know lots of representations of $\mathfrak{sl}_2$: Consider $L(n)$, all degree $n$ homogeneous polynomials in $X, Y$ where $e = X\frac{\partial}{\partial Y}$, $f = Y\frac{\partial}{\partial X}$, and $h = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$. One can easily check these are representations, and they are certainly simple:

As shown in the figure below, applying $e$ a finite number of times from $Y^n$ eventually gives some multiple of $X^n$, and vice versa with $f$, so every element of L(n) is reached by an element of $\mathfrak{sl}_2$. So any submodule that contains $Y^n$ must contain all of $L(n)$, and similarly any submodule containing $X^n$ must contain all of $L(n)$. If a submodule contained some nontrivial subset excluding both $X^n$ and $Y^n$, for any element $x$ we can take $f^k x \neq 0$ such that $f^{k+1}x = 0$. This necessarily gives us some multiple of $Y^n$, and applying $e$ successively recovers $L(n)$.

$$\begin{tikzcd}
Y^3 \arrow[r, bend left, "3"] \arrow[loop below, "-3"] & XY^2 \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] \arrow[loop below, "-1"] & X^2Y \arrow[l, bend left, "2"] \arrow[r, bend left, "1"] \arrow[loop below, "1"] & X^3 \arrow[l, bend left, "3"] \arrow[loop below, "3"]
\end{tikzcd}$$

Above is an pictoral depiction of $L(3)$, a simple representation of $\mathfrak{sl}_2$. The arrows pointing to the right denote scaling after applying $e = X\frac{\partial}{\partial Y}$, similarly the arrows pointing to the left denote scaling after applying $f = Y\frac{\partial}{\partial X}$, and the loops are $h$-eigenvalues.

We need three more quick definitions before we can begin proving the theorem:

**Definition 3.3.** A vector $v_\lambda$ in a representation is said to be of *highest weight* if it is an $h$-eigenvector with the largest (positive) eigenvalue.

**Definition 3.4.** The *universal enveloping algebra* $U(\mathfrak{g})$ is the algebra consisting of all words of elements in $\mathfrak{g}$ modulo the commutator relations $[X, Y] = XY - YX$. More abstractly, for any associative algebra $A$, $U(\mathfrak{g})$ is the unique algebra satisfying

$$\mathrm{Hom}_{Alg}(U(\mathfrak{g}), A) = \mathrm{Hom}_{Lie}(\mathfrak{g}, A)$$

For example, in the case of $\mathfrak{sl}_2$, $e^2 f$ is in the universal enveloping algebra, even if it is not itself an element of $\mathfrak{sl}_2$.

**Definition 3.5.** The *Casimir* $\Omega \in \mathrm{Z}(U(\mathfrak{g}))$ Is defined as $\Omega = 2fe + \frac{h(h+2)}{2}$.

The Casimir is central in the universal enveloping algebra. This is not too hard to see, but requires some computation that we will avoid.

Now let us prove Theorem 3.1.

*Proof.* Our task is two-fold: First, we must show that every irreducible $\mathfrak{sl}_2$ representation $L$ is isomorphic to $L(n)$ for some $n$. Then we must show that every finite-dimensional $\mathfrak{sl}_2$ representation can be decomposed as a direct sum of $L(n)$'s.

Let $L$ be an irreducible $\mathfrak{sl}_2$ representation, and let $v \in L$ be an $h$-eigenvector.

Now consider the collection $\{v, ev, e^2v, \ldots\}$. By definition, $hv = \lambda v$ for some scalar $\lambda$.

Then

$$\begin{aligned}
h(ev) =& (eh + [h, e])v \\
=& \lambda ev + 2ev \\
=& (\lambda + 2)ev
\end{aligned}$$

Similarly, $he^2v = (\lambda + 4)e^2v$. Since $\{v, ev, e^2v, \ldots\}$ are nonzero $h$-eigenvectors with different eigenvalues, they are linearly independent, but we are assuming $L$ is a finite-dimensional vector space. So there must exist $k : e^k v \neq 0, e^{k+1}v = 0$.

Set $u = e^k v$ to be our highest weight vector, so $eu = 0$, $hu = nu$ for some $n$.

Claim: $n \in \mathbb{N}$.

*Proof.* To prove this, we need a quick lemma.

**Lemma 3.6.** *For $u \in L$ highest weight vector, we have*

$$ef^k u = k(n-k+1)f^{k-1}u$$

We omit the proof since it is direct computation. □

So then we see that $\{f^n u, f^{n-1}u, \ldots, fu, u\}$ is a set of linearly independent vectors in $L$. Since $L$ is finite dimensional, there exists $r$ for which $ef^{r+1} = 0 = (r+1)(n-r)f^{r+1}u$ or $n = r$, so $n$ is a positive integer.

This proves that every simple representation $L$ is isomorphic to $L(n)$ for some $n$.

Recall from before that the Casimir $\Omega$ is a central element. Hence the generalized eigenspaces $V^\lambda = \ker(\Omega - \lambda)^N \subset V$ are $\mathfrak{sl}_2$ subrepresentations. This is because if $(\Omega - \lambda)^N v = 0$, then $(\Omega - \lambda)^N ev = e(\Omega - \lambda)^N v = 0$ (as with $f$ and $h$), showing that each generalized eigenspace is closed under the $\mathfrak{sl}_2$ operations.

Using Schur's Lemma (which we will not prove), $\Omega$ acts on $L(n)$ by a scalar given by $x = \frac{n(n+2)}{2}$ So $\ker(\Omega - x) = L(n)$.

For any finite dimensional vector space $V$, using Jordan Canonical Form we can write $V$ as a direct sum of generalized eigenspaces. Write $V = \bigoplus V^x = \bigoplus V^{n(n+2)/2}$.

Claim: $h$ acts by exactly $n$ on $\ker e \subset V^{n(n+2)/2}$. Loosely speaking, this will tell us that $h$ does not map elements in one copy of $L(n)$ into another, and thus our decomposition is actually a direct sum of $L(n)$'s.

To prove this, another formula we will make use of is the following:

$$ef^{n+1} = f^{n+1}e + (n+1)f^n(h-n)$$

If $v \in \ker(e)$, then since $v \in V^{n(n+2)/2}$, we have

$$0 = ef^{n+1}v = f^{n+1}ev + (n+1)f^n(h-n)v = 0 + (n+1)f^n(h-n)v$$

Which tells us that $h = n$, precisely what we wanted.

Notably, we needed two properties of $V^{n(n+2)/2}$: First, that $f^{k+1}v = 0$, and second that $f^k v \neq 0$. These results are proved in detail in [3].

This finishes.

□

## 4. Finite Dimensional Representations of sl(3) with a generalizing approach

For $\mathfrak{sl}_3$, the approach is somewhat more complicated. First, observe that $\mathfrak{sl}_3$ is again the Lie Algebra of 3x3 matrices whose trace is 0. We begin by setting up the preliminaries to state the main theorem, and then give key ideas regarding its proof.

**Definition 4.1.** The torus $\mathfrak{t}$ is the maximal abelian subalgebra of $\mathfrak{sl}_3$

In our case $\mathfrak{t}$ is simply diagonal matrices of trace 0, hence they are spanned by matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Note that this requires a choice! However, every maximal semisimple commutative subalgebra is some conjugate of $\mathfrak{t}$, although we will not prove this.

The $\mathfrak{sl}_3$ analog of the $h$-eigenvalue $n$, which was a scalar in $\mathfrak{sl}_2$, is now an element of the dual torus. Let $\epsilon_i \in \mathfrak{t}^*$ be given by $\epsilon_i(X)$ be the $i$th entry $X_i$ of any diagonal matrix $X$. So then it is immediately obvious that $\mathfrak{t}^* =< \epsilon_1, \epsilon_2, \epsilon_3 > /(\epsilon_1 + \epsilon_2 + \epsilon_3)$.

One can show that for the matrix $E_{i,j}$ consisting of a 1 in the $i$th row, $j$th column, and 0 everywhere else, $[X, E_{i,j}] = (\epsilon_i - \epsilon_j)(X) * E_{i,j}$. So then the roots $R = \epsilon_i - \epsilon_j$ for $i \neq j$.

The root space decomposition is then
$\mathfrak{sl}_3 = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{sl}(3)_\alpha = \mathfrak{t} + \bigoplus_{\epsilon_i-\epsilon_j\in R} \mathfrak{sl}(3)_{\epsilon_i-\epsilon_j}$
where $\mathfrak{sl}(3)_\alpha = \{x \in \mathfrak{sl}_3 | \forall t \in \mathfrak{t}, [t,x] = \alpha(t)x\}$.

Next, we observe that $\mathfrak{t}$ has an inner product: namely, for any matrices $A, B \in \mathfrak{t}$, $\langle a, b\rangle = \mathrm{tr}(AB)$. This induces an inner product on $\mathfrak{t}^*$, and we can normalize this in such a way that $\langle \alpha, \alpha\rangle = 2$.

**Definition 4.2.** The *weight lattice* $P$ is defined as

$$P = \{\lambda \in \mathfrak{t}^* | \forall \alpha \in R, \langle \lambda, \alpha\rangle \in \mathbb{Z}\}$$

**Definition 4.3.** The *dominant cone* is defined as

$$\Lambda^{dom} = \{\lambda \in \mathfrak{t}^* | \text{for } \alpha = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}, \langle \lambda, \alpha\rangle \in \mathbb{Z}_{\geq 0}\}$$

We are now ready to state the big theorem, stated for $\mathfrak{sl}_3$ but true without too much additional work for $\mathfrak{sl}_n$:

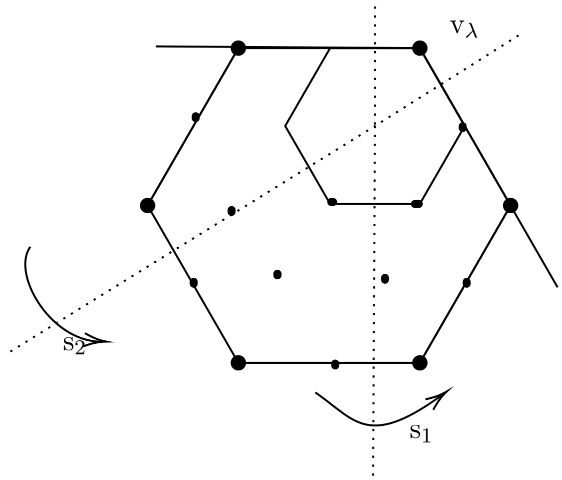
**Theorem 4.4.** *Every finite dimensional representation of $\mathfrak{sl}_3$ can be decomposed as a direct sum of simple representations, which correspond to unique $\lambda \in \Lambda^{dom}$, which are the weights of the highest weight vectors $v_\lambda$.*

We will give a brief sketch our progress on the proof:

Like with $\mathfrak{sl}_2$, we need to prove two things: first, that every finite dimensional irreducible representation of $\mathfrak{sl}_3$ contains a unique highest weight vector $v_\lambda$. To prove this, we look at what are called $\mathfrak{sl}_2$-triples, which are sets of elements $e, f, h \in \mathfrak{sl}_3$ which satisfy the $\mathfrak{sl}_2$ commutation relations. This will give us the result for basis elements of the torus, and it suffices to use linearity to get an arbitrary $h$.

Then we need to show that for each dominant weight $\lambda$ there exists a finite dimensional representation $L(\lambda)$ with highest weight $\lambda$. To do this, we consider Verma modules, $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}v_\lambda$, where we are taking the tensor product of the universal enveloping algebra with a one-dimensional vector space spanned by the highest weight vector over $U(\mathfrak{b})$ where $\mathfrak{b}$ consists of the torus and positive roots. One can prove that $M(\lambda)$ has a basis spanned by $f_1^{n_1} f_2^{n_2} f_3^{n_3} \otimes v_\lambda$ for all $n \in \mathbb{N}$, where $f_1 = E_{2,1}$, $f_2 = E_{3,2}$, and $f_3 = E_{3,1}$. $M(\lambda)$ is infinite dimensional, so we must quotient out by some relation to obtain finite-dimensional representations. Construct $L(\lambda) = \frac{M(\lambda)}{\left(f_1^{\langle\lambda,\alpha_1\rangle+1} M(\lambda) + f_2^{\langle\lambda,\alpha_2\rangle+1} M(\lambda)\right)}$
To show this is finite dimensional, we use $\mathfrak{sl}_2$ triples once again to effectively project onto horizontal lines in the weight lattice to get a direct sum of representation of $\mathfrak{sl}_2$, and use the fact that finite dimensional representations of $\mathfrak{sl}_2$ are invariant under reflection to force the quotient to remain in a finite region and conclude that our representation of $\mathfrak{sl}_3$, $L(\lambda)$ is finite dimensional. (our $f_i$ act nilpotently). The character of the quotient is invariant under the simple reflections, which generate what is known as the Weyl Group. See Figure **??** for a geometric picture.

There is still much left to prove; this establishes that the quotients $L(\lambda)$ correspond to particular points in the dominant cone $\Lambda^{dom}$, but we have not yet proven that these are unique, nor have we shown that this free construction generates all possible finite dimensional representations. This would be relegated to future work if our directed reading program continued, or it may be covered in a course such as Math 261.




The diagram above shows the hexagonal weight lattice that defines $\mathfrak{sl}_3$. In the top right, the point labeled $v_\lambda$ is the highest weight vector, and all points in the lattice in the bottom-left third of the lattice separated by the dark extending lines constitute $M(\lambda)$. To construct $L(\lambda)$, the equivalence relations result in reflectional symmetries $s_1$ and $s_2$ which force the resulting portion of the lattice to be finite dimensional.

## 5. A Brief Digression on Schur-Weyl Duality

We conclude with some brief remarks in the interest of space and time.

Using the notation we have developed for $\mathfrak{sl}_2$, consider $L(1) = \mathbb{C}^2 = V$. We want to consider $V \otimes V$. This tensor product is spanned by four basis elements: $v_1 \otimes v_1$, $v_1 \otimes v_{-1}$, $v_{-1} \otimes v_1$, and $v_{-1} \otimes v_{-1}$. Clearly, this is a finite dimensional representation, and hence it can be decomposed as a direct sum of simple representations. Applying $e$, $f$, and $h$ shows the correct basis to take is actually $v_1 \otimes v_1$, $v_1 \otimes v_{-1} + v_{-1} \otimes v_1$, $v_{-1} \otimes v_{-1}$, and $v_1 \otimes v_{-1} - v_{-1} \otimes v_1$. This decomposes as $L(2) \oplus L(0)$. $\mathfrak{sl}_2$ has a nontrivial action on $L(2)$ but has a trivial action on $L(0)$, whereas the symmetric group $S_2$ acts trivially on $L(2)$ but acts by flipping the sign on $L(0)$. Hence it can be expressed as $S^2V \oplus \Lambda^2 V$, the direct sum of a symmetric and antisymmetric component.

More generally, using the formula $h(v \otimes w) = hv \otimes w + v \otimes hw$, we find that the symmetric and antisymmetric components of the direct sum alternate. For instance, $L(3) \otimes L(3) = (L(6) \oplus L(2)) \oplus (L(4) \oplus L(0)) = S^2L(3) + \Lambda^2 L(3)$.


## Acknowledgements

I would like to thank Jeremy Taylor for teaching the material and being receptive to all of my questions! Thank you to the staff behind the Directed Reading Program for organization.


## References


1. Roger W Carter, Graeme Segal, and Ian Grant Macdonald, *Lectures on lie groups and lie algebras*, Cambridge University Press, 1995.
2. William Fulton and Joe Harris, *Representation theory: a first course*, vol. 129, Springer Science & Business Media, 2013.
3. Ian Grojnowski and typeset by Elena Yudovina, *Introduction to lie algebras and their representations part iii michaelmas 2010.*



Department of Mathematics, University of California, Berkeley,
*Email address*: `arenmatinian@berkeley.edu`