Gravitational instantons and K3 surfaces

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Gravitational instantons

Definition

(X^4, g) is a *gravitational instanton* if g is complete, $\mathrm{Hol}(g) \subset \mathrm{SU}(2)$, and $\int_X |Rm|^2 dV_g < \infty$.

- $\mathrm{Hol}(g) \subset \mathrm{SU}(2)$ called "Calabi-Yau" $\iff$ X is a Kähler manifold and the canonical bundle K_X trivial.
- $\mathrm{Hol}(g) \subset \mathrm{Sp}(1)$ called "hyperkähler" $\iff$ $\exists$ complex structures I, J, K such that g is Kähler with respect to

$$aI + bJ + cK, \ a^2 + b^2 + c^2 = 1,$$

an S^2s worth of complex structures.

- Since $\mathrm{Sp}(1) = \mathrm{SU}(2)$ these are equivalent.

Ricci-flatness

Why "gravitational instanton"?

- Calabi-Yau $\Longrightarrow \exists$ parallel $(2,0)$-form $\Omega = \omega_J + i\omega_K$.
- Since $\Lambda^2_+ = \mathbb{R}\omega \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \Longrightarrow \Lambda^2_+$ is flat.
- The curvature of Λ^2_+ is given by

$$0 = \mathcal{R}(\Lambda^2_+) = \big(W^+ + (R/12)Id \qquad Ric - (R/4)g \big)$$

  In particular, $Ric \equiv 0, W^+ \equiv 0$, and $*\mathcal{R}_g = -\mathcal{R}_g *$.

- The "gravitational" terminology arises from the analogy with general relativity, and the "instanton" terminology arises from the analogy with ASD Yang-Mills connections.

Ricci-flatness

A few remarks:

- Ricci-flat Kähler: still have $*\mathcal{R}_X = -\mathcal{R}_X *$, but more general.
- Ricci-flat Kähler and $\pi_1(X) = \{e\} \Longrightarrow$ hyperkähler.
- $\exists$ isometric quotients of hyperkähler which are "only" Ricci-flat Kähler (e.g., finite quotients of ALE A_k metrics).
- There are some hyperkähler examples which are not simply connected (e.g., $\mathbb{R} \times T^3$, Atiyah-Hitchin on $S^4 \setminus \mathbb{RP}^2$, ALH* on $\mathbb{CP}^2 \setminus T^2$ or $S^2 \times S^2 \setminus T^2$).
- Could require only the Ricci-flat condition. But there are not many known examples which are not Kähler (e.g., Euclidean Schwarzschild or Euclidean Kerr on $\mathbb{R}^2 \times S^2$, Page's Taub-bolt on $\mathbb{CP}^2 \setminus \{p\} = \mathcal{O}_{\mathbb{P}^1}(-1)$, Biquard-Minerbe ALH example).

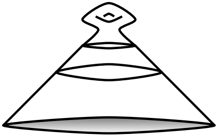
ALE gravitational instantons

Definition

(X, g) is ALE if

$$g = g_{\mathbb{R}^4/\Gamma} + O(r^{-\delta})$$

as $r \to \infty$, $\Gamma \subset \mathrm{SO}(4)$.



ALE gravitational instantons

Remarks:

- Example: Eguchi-Hanson metric, $|\Gamma| = 2$, $X = T^*S^2$. More generally, examples of type A_k (multi-Eguchi-Hanson), D_k, E_6, E_7, E_8 (classified by Kronheimer).
- $Vol(B(p,r)) \sim r^4$.
- Tangent cone at infinity: $\mathbb{R}^4/\Gamma$.
- $|Rm| = O(r^{-6})$, can choose $\delta \geq 4$ (Bando-Kasue-Nakajima).

ALF gravitational instantons

Definition

(X, g) is ALF if

$$g = dr^2 + r^2(\pi^* g_{S^2}) + \theta^2 + O(r^{-\delta})$$

as $r \to \infty$, where $\pi : S^3 \to S^2$ is the Hopf fibration, and θ is a connection form. Can also take quotients by $\mathbb{Z}_k$ in the fiber direction, and also replace S^2 by $\mathbb{RP}^2$.

- Examples: Taub-Nut metric on $\mathbb{C}^2$ (see LeBrun). More generally, examples of type ALF-A_k (multi-Taub-NUT) and ALF-D_k (Cherkis-Hitchin-Kapustin).
- $Vol(B(p, r)) \sim r^3$, tangent cone at infinity: $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm 1\}$.
- $|Rm| = O(r^{-3})$, can choose $\delta < 1$ (Minerbe, Chen-Chen). Classified by Chen-Chen.

Elliptic Fibrations

Elliptic surface: $\pi : X \to C$, holomorphic, with generic fiber genus 1. Let $S = \{p_1, \ldots, p_k\}$ be the image of the singular fibers. Kodaira's fundamental invariants:

- Monodromy: $\rho : \pi_1(C \setminus S) \to SL(2, \mathbb{Z})$.
- Periods: $\tau : C \setminus S \to \mathbb{H}$ (multivalued), Functional invariant: $j : C \setminus S \to \mathbb{P}^1$. (Determines the complex structure of smooth fibers).
- Compatibility: along a closed arc $\gamma : S^1 \to C$, any single-valued lifting of τ transforms by the corresponding monodromy matrix, up to ± 1.

Singular fibers

We consider only the cases there are no multiple fibers, and all fibers are minimal. Singular fibers are classified by Kodaira:

- $\mathrm{I}_b, b \geq 1, \rho(\gamma) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, j$ has a pole of order b at p.
- $\mathrm{I}_b^*, b \geq 1, \rho(\gamma) = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}, j$ has a pole of order b at p.
- Finite monodromy: $I_0^*, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$, $j(0) = \alpha, 0, 1, 0, 0, 1, 0$ respectively, $\alpha \in \mathbb{C}$.

ALG gravitational instantons

Define

$$g_{\beta,\tau} = g_{Euc} + g_{\tau}$$

on $C(0, 2\pi\beta) \times T^2$, where

$$C(0, 2\pi\beta) = \{z \in \mathbb{C}^* \mid 0 < Arg(z) < 2\pi\beta\},$$

g_τ is a flat metric on T^2, parametrized by $\tau \in \mathbb{H}$.

Definition

(X, g) is ALG if on a dense subset,

$$g = g_{\beta,\tau} + O(r^{-\delta}), \quad r \to \infty.$$

ALG gravitational instantons

Remarks:

- Examples arise from any rational elliptic surface

$$\Sigma = Bl_{p_1,\dots,p_9}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^1,$$

and $X = \Sigma \setminus D$, where D is a finite monodromy fiber (Hein).

Type	I_0^*	II^*	III^*	IV^*	II	III	IV
β	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$
τ	any	ζ_3	i	ζ_3	ζ_3	i	ζ_3

- The sector in the ALG definition is "closed up" by the monodromy around the singular fiber.

- $Vol(B(p,r)) \sim r^2$, tangent cone at infinity: $C(2\pi\beta)$.

- $Rm = O(r^{-2-\epsilon})$, classified by Chen-Chen.

ALG* gravitational instantons

Let $b = 2\nu$, $V = \frac{b}{2\pi}\log(r)$, and

$$g_b = V(dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + V^{-1} d\theta_3^2$$

where $\{d\theta_1, d\theta_2, d\theta_3\}$ is a left-invariant coframe on a Heisenberg nilmanifold Nil_b^3, which is an S^1 bundle over T^2 of degree b,

$$S^1 \longrightarrow Nil_b^3 \xrightarrow{\pi} T^2 ,$$

such that $d\theta_3$ is a connection form satisfying $d\theta_3 = \frac{b}{2\pi} d\theta_1 \wedge d\theta_2$.

Let $\tilde{g}_\nu$ denote the metric on the $\mathbb{Z}_2$ quotient by the action

$$(\theta_1, \theta_2, \theta_3) \mapsto (\theta_1 + \pi, -\theta_2, -\theta_3).$$

Cross section is an *infranilmanifold*.

ALG* gravitational instantons

Definition

(X, g) is ALG* if $g = \tilde{g}_\nu + O(r^{-\delta}), r \to \infty$.

- Examples arise from rational elliptic surfaces Σ: $X = \Sigma \setminus D$, where D is a singular fiber of type I_ν^*, $1 \leq \nu \leq 4$ (Hein).
- $Vol(B(p, r)) \sim r^2$.
- Tangent cone at infinity: $\mathbb{R}^2/\{\pm 1\}$.
- $Rm = O(r^{-2}(\log(r))^{-1})$, as $r \to \infty$.

ALH gravitational instantons

Definition

(X, g) is ALH if

$$g = dr^2 + g_{T^3} + O(r^{-\epsilon}),\ r \to \infty,$$

where g_{T^3} is a flat metric on T^3.

- Examples arise from rational elliptic surfaces Σ: $X = \Sigma \setminus T^2$, where T^2 is a smooth fiber (Hein).
- $Vol(B(p, r)) \sim r$.
- Tangent cone at infinity: $\mathbb{R}_+$ (unless $X = \mathbb{R} \times T^3$).
- $|Rm| = O(e^{-\delta r})$, classified by Chen-Chen.

ALH* gravitational instantons

Define

$$g_b = dr^2 + r^{2/3}\pi^* g_{T^2} + r^{-2/3}\theta_b^2,$$

where θ_b is a connection form on Nil_b^3, which is an S^1 bundle over T^2 of degree b:

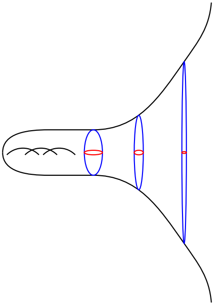
$$S^1 \longrightarrow Nil_b^3 \xrightarrow{\pi} T^2$$

satisfying $d\theta_b = 2\pi b A^{-1} dV_{T^2}$.

Definition

(X, g) is ALH^* if $g = g_b + O(e^{-\delta r^{2/3}})$, as $r \to \infty$.

ALH* gravitational instantons



The red circles represent the S^1 fibers, the blue curves represent the T^2s. In terms of distance to a basepoint,

$$\mathrm{diam}(Nil_b^3(r)) \sim r^{1/3}, \quad \mathrm{diam}(S^1(r)) \sim r^{-1/3}.$$

ALH* gravitational instantons

Remarks:

- Examples arise from rational elliptic surfaces Σ: $X = \Sigma \setminus D$, where D is a singular fiber of type I_b, $1 \leq b \leq 9$.
- $Vol(B(p,r)) = O(r^{4/3})$ as $r \to \infty$.
- Tangent cone at infinity: $\mathbb{R}_+$.
- $|Rm| = O(r^{-2})$ as $r \to \infty$, but not any better.

K3 surfaces

If X is *compact* complex surface which is simply connected and has $c_1(X) = 0$ then X is diffeomorphic to

$$K3 = \{z_0^4 + z_1^4 + z_2^4 + z_3^2 = 0\} \subset \mathbb{P}^3.$$

By Yau's Theorem, every Kähler class for every complex structure admits hyperkähler metrics.

$\dim_{\mathbb{R}}(\mathcal{M}(K3)) = 58$, 40 for complex structures, 20 for Kähler classes, but subtract 2 since metric is hyperkähler.

It is known that

$$\mathcal{M}(K3) = \Gamma \backslash \mathrm{SO}_{\circ}(3,19)/\mathrm{SO}(3) \times \mathrm{SO}(19),$$

where Γ is a discrete arithmetic subgroup. (Note this description includes orbifold K3 Einstein metrics).

General theory

What happens near the boundary?

- $Ric(g_j) = 0 \implies$ Gromov-Hausdorff limit.
- Singularity formation $\implies$ curvature blows up.
- Bubbling phenomena: non-collapsed rescaled limits are *gravitational instantons*.
- Volume non-collapsing: $Vol(B_{p_j}(1)) > v_0 > 0 \implies$ orbifold limit.
- Volume collapsing $Vol(B_{p_j}(1)) \to 0 \implies$ lower-dimensional limit.

Theorem (Cheeger-Tian)

Sequence collapses with uniformly bounded curvature away from finitely many points.

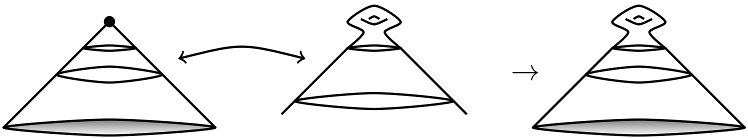
ALE bubbles

Recall: (X, g) is ALE if

$$g = g_{\mathbb{R}^4/\Gamma} + O(r^{-\delta})$$

as $r \to \infty$, $\Gamma \subset \mathrm{SO}(4)$.

Kummer surface: 4-dim limit $= T^4/\mathbb{Z}_2$, with flat metric. At 16 singular points, Eguchi-Hanson metric on $\mathcal{O}_{\mathbb{P}^1}(-2)$ bubbles off.



ALF bubbles

Recall: (X, g) is ALF if

$$g = dr^2 + r^2(\pi^* g_{S^2}) + \theta^2 + O(r^{-\delta})$$

as $r \to \infty$, where $\pi : S^3 \to S^2$ is the Hopf fibration, and θ is a connection form (or $\mathbb{RP}^2$).

Foscolo: modified Kummer construction, 3-dim limit $= T^3/\mathbb{Z}_2$, with flat metric. At 8 singular points, ALF D_2 metrics bubble off.

ALH bubbles

Recall: (X, g) is ALH if

$$g = dr^2 + g_{T^3} + O(e^{-\delta r}).$$

as $r \to \infty$, with $Vol(B(p,r)) \sim r$.

Chen-Chen: 1-dim limit $= [0,1]$. Singular points at $0, 1$. Interior: collapse with uniformly bounded curvature, uniform shrinking of flat T^3.

Produced by gluing together 2 ALH factors with a long cylindrical region in between, using earlier ideas of Kovalev-Singer, Floer.

Tian-Yau metrics

Let DP_b be a degree $1 \leq b \leq 9$ del Pezzo surface. Let $T^2 \subset DP_b$ be a smooth anticanonical divisor.

Theorem (Tian-Yau)

$X_b = DP_b \setminus T^2$ admits a complete Ricci-flat Kähler metric, which is asymptotic to a Calabi ansatz metric on a punctured disc bundle in N_{T^2}.

Solution of the form $\omega_g = \frac{i}{2\pi}\Big\{\partial\overline{\partial}(-\log\|S\|^2)^{\frac{3}{2}} + \partial\overline{\partial}\phi\Big\}$.

We would like to "glue" two of these spaces together, but the asymptotic geometry is not cylindrical: need to find appropriate neck region.

Tian-Yau metrics are ALH*

Theorem (Hein-Sun-V-Zhang)

A Tian-Yau metric on $X_b = DP_b \setminus T^2$ is ALH, with*

$$g = g_b + O(e^{-\delta r^{2/3}})$$

as $r \to \infty$, for some $\delta > 0$.

The proof relies on finding improved asymptotics for the complex structure, and then using techniques in Hein's thesis and Tian-Yau.

Hein-Sun-V-Zhang

Theorem (HSVZ)

Given integers $1 \leq b_{\pm} \leq 9$*, there is a family of hyperkähler metrics* g_β *on a K3 surface which collapse to an interval* $[0, 1]$*,*

$$(K3, g_\beta) \xrightarrow{GH} ([0, 1], dt^2), \;\; \beta \to \infty,$$

with the following properties:

- *The "bubbles" at the endpoints are Tian-Yau metrics on del Pezzo surfaces of degree* $b_{\pm}$ *minus an anticanonical elliptic curve.*
- *In the interior region, there are* $b_+ + b_-$ *Taub-NUT bubbles.*

K3, illustrated

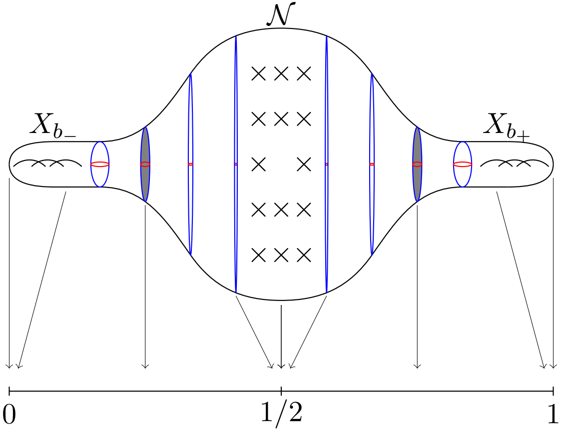


Figure: The vertical arrows represent collapsing to a one-dimensional interval. The red circles represent the S^1 fibers, the blue curves represent the T^2 directions, and the $\times$s are Taub-NUT metrics.

Hein-Sun-V-Zhang

Remarks on HSVZ:

- Neck region is produced using the Gibbons-Hawking ansatz over $T^2 \times \mathbb{R}$, and a harmonic function which is asymptotically linear in z on each end, and with $b_+ + b_-$ monopole points in the interior.
- Produced using technique of gluing hyperkähler triples, introduced by Donaldson, and also used in Foscolo's work.
- In the regular collapsing regions, nilmanifold collapsing occurs: the T^2 directions and the S^1 directions shrink at different rates

$$\text{diam}(Nil^3_b) \sim \beta^{-1}, \quad \text{diam}(S^1) \sim \beta^{-2}.$$

Elliptic K3 surfaces

Remarks:

- Gross-Wilson: Case of elliptic K3 with 24 fibers of type I_1 (nodal cubics). 2-dim limit $= S^2$. Away from 24 singular points, sequence collapses with uniformly bounded curvature, with T^2-fibers being uniformly scaled down.
  Gross-Tosatti-Zhang: any elliptic K3, GH limit is S^2.
  Odaka-Oshima made further progress. Bubbles?
- In joint work with Gao Chen and Ruobing Zhang, we show it is possible to generalize Gross-Wilson to any elliptically fibered K3 surface (24 I_1 fibers is the generic case) AND understand the behavior near the singular fibers. In particular, in the Gross-Wilson case, we can show that at the 24 singular points, Taub-NUT ALF metrics bubble off.

Chen-V-Zhang

Theorem (Chen-V-Zhang)

For any elliptic K3 surface $\pi : X \to \mathbb{P}^1$, there exists a family of Ricci-flat Kähler metrics g_ϵ on X such that:

- *The area of a regular fiber is ϵ, and $(X, g_\epsilon) \xrightarrow{\mathrm{GH}} (S^2, g_{McLean})$ as $\epsilon \to 0$.*
- *Near singular fibers with finite monodromy, bubbles are ALG gravitational instantons.*
- *Near singular fibers with infinite monodromy, there are b Taub-NUT bubbles in the I_b case and b Taub-NUT bubbles plus 4 Eguchi-Hanson bubbles in the I_b^* case.*

Greene-Shapere-Vafa-Yau semi-flat metric

Let $\pi : X \to \mathbb{P}^1$ be an elliptic K3 surface with a holomorphic section.

Fix a non-vanishing holomorphic 2-form Ω on X, for any small enough disc $E \subset \mathbb{P}^1 \setminus S$, for any fixed holomorphic coordinate y on E, there exists a unique local coordinate $x \in \mathbb{C}/(\mathbb{Z}\tau_1(y) \oplus \mathbb{Z}\tau_2(y))$ such that $\Omega = dx \wedge dy$ locally on $X|_E$.

Write

$$x = x_1\tau_1(y) + x_2\tau_2(y),\ \ x_1, x_2 \in \mathbb{R}/\mathbb{Z},$$

and define

$$\omega_\delta^{\mathrm{sf}} = \delta^2 \cdot dx_1 \wedge dx_2 + \underbrace{\frac{\sqrt{-1}}{2} \cdot \mathrm{Im}(\bar{\tau}_1\tau_2) dy \wedge d\bar{y}}_{g_{McLean}}.$$

Resolving the singularities

The semi-flat metric is singular near the singular fibers. To resolve:

- Finite monodromy fibers: the asymptotics of the *dual* isotrivial ALG metrics agree with the asymptotics of the semi-flat metric near the fibers with finite monodromy, so we can glue these onto the semi-flat metric near these fibers.
- Infinite monodromy fibers: glue in an incomplete "multi-Ooguri-Vafa metric" in the I_b case, or a multi-Ooguri-Vafa metric with $2b$ monopole points modulo a $\mathbb{Z}_2$ action in the I_b^* case (and 4 Eguchi-Hanson metrics to resolve the 4 ODP).

Work in progress

Theorem (Chen-V-Zhang)

For any elliptic K3 surface $\pi : X \to \mathbb{P}^1$, there exists a family of Ricci-flat Kähler metrics g_ϵ on X such that:

- *Near singular fibers of of type $I_b^*, 0 \leq b \leq 14$, given any integer $0 \leq \nu \leq 4$ there can be an ALG_ν^* gravitational instanton bubble plus $b + \nu$ Taub-NUT bubbles.*

We cannot do this with a fixed complex structure. The idea is similar to HSVZ: use a Gibbons-Hawking ansatz over $\mathbb{R}^2 \times S^1$ with a suitable harmonic function, we can construct a neck region which interpolates between the semi-flat metric near the I_b^* fiber and the ALG_ν^* bubble.

Summary

In summary, the known GH limits and bubbles arising from sequences of Ricci-flat metrics on the K3 surface:

Type	Vol(B(p,r))	Case	G-H limit
ALE	$\sim r^4$	Kummer	$T^4/\mathbb{Z}_2$
ALF	$\sim r^3$	Foscolo	$T^3/\mathbb{Z}_2$
ALG	$\sim r^2$	Chen-V-Zhang	S^2
ALG$^*_\nu, 1 \leq \nu \leq 4$	$\sim r^2$	Chen-V-Zhang	S^2
ALH	$\sim r$	Chen-Chen	$[0, 1]$
ALH$^*_b, 1 \leq b \leq 9$	$\sim r^{\frac{4}{3}}$	HSVZ	$[0, 1]$

Question

Are there any other possible collapsed GH limits?

Question

Are there any other possible gravitational instanton bubbles?

End

Thank you for your attention.