Universality for diffusions interacting through a random matrix

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Joint works with Reza Gheissari, Eyal Lubetzky and Ofer Zeitouni

Overview

- Motivation: spin-glass models, Langevin dynamics
- Limiting dynamics: Gaussian disorder
- Universality: the challenge for dynamics
- Combining Girsanov and Lindeberg **(w. Lubetzky & Zeitouni)**
- Stochastic Taylor expansion **(w. Gheissari)**

Spin-glass models

Random Gibbs measures on $\mathbb{R}^N$ at inverse temperature $\beta > 0$,

$$\nu^N_{\beta,\mathbf{J}}(A) = Z^{-1}_{\beta,\mathbf{J}} \int_A e^{\beta H_{\mathbf{J}}(\mathbf{x})} e^{-2U(\mathbf{x})} d\mathbf{x}\,, \qquad A \subset \mathbb{R}^N\,,$$

with random $H_{\mathbf{J}} : \mathbb{R}^N \to \mathbb{R}$ and normalizing constant $Z_{\beta,\mathbf{J}} = \int e^{\beta H_{\mathbf{J}}(\mathbf{x}) - 2U(\mathbf{x})} d\mathbf{x}$.
Potential $U(\mathbf{x})$ tunes the support (e.g. near the hypercube $\{\pm 1\}^N \subset \mathbb{S}^N$).

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Mixed p-spin models: H_{J} a centered Gaussian function

$$\mathrm{Cov}\big(H_{\mathbf{J}}(\mathbf{x}), H_{\mathbf{J}}(\mathbf{y})\big) = N\xi\big(N^{-1}\langle \mathbf{x}, \mathbf{y}\rangle\big)\,, \quad \xi(r) := \sum_{p \le m} b_p^2 r^p$$

$m = 2$ is Sherrington-Kirpatrick (SK) model; $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle$, Euclidean norm.

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Spin-glasses (on $\{\pm 1\}^N$), are toy models of diluted magnetic systems with random interactions (examples of disordered mean-field models).
Spherical (on $\mathbb{S}^N$), often serving as further (mathematical) simplification.

Much recent progress in understanding the asymptotic $N \to \infty$ of $\nu^N_{\beta,\mathbf{J}}(\cdot)$ starting with $F_\beta = \lim N^{-1} \log Z_{\beta,\mathbf{J}}$ (Talagrand (06'), Panchenko (13'), ...).

Langevin dynamics for soft spins

Langevin particles $\mathbf{x}_t = (x_t^{(i)})_{1 \le i \le N} \in \mathbb{R}^N$, solution of diffusion

$$d\mathbf{x}_t = \beta \nabla H_{\mathbf{J}}(\mathbf{x}_t)dt - \nabla U(\mathbf{x}_t)dt + d\mathbf{B}_t$$

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Langevin dynamics is invariant for (random) Gibbs measure

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$U_\star(r) = \infty$ outside $(-\mathfrak{s}, \mathfrak{s})$, minimal at $r = \pm 1$ (supported near $\{\pm 1\}^N$).

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For $\mathbf{x}_0$ of i.i.d. entries, the (soft binary SK, Langevin) diffusion

$$dx_t^{(i)} = -U_\star'(x_t^{(i)})dt + \frac{\beta}{\sqrt{N}}\sum_{j=1}^N J_{ij}x_t^{(j)}dt + dB_t^{(i)}$$

predicted to have exponential in N relaxation time when $\beta \gg 1$
$\Longrightarrow$ Experiments can only observe the system out of equilibrium.

Limiting dynamics: Gaussian disorder, binary-spins

Consider empirical measures of particle trajectories in $[0, T]$,

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_\cdot^{(i)}} \in \mathcal{M}_1(C([0,T]))\,,$$

for $U_\star(r) \to \infty$ as $|r| \to \mathfrak{s}$, denoting by $\mathbb{P}_\beta$ the law of interacting diffusions

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BenArous-Guionnet (95'), Guionnet (97') (symmetric J): as $N \to \infty$
$\mu_N \overset{a.s.}{\to} \mu_\star$ law of self-consistent non-Markovian single-spin diffusion.
(predicted by Cristiani-Sampolinski (87'); [CS87] and
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Key: Explicit Gaussian computation of $\Gamma_N(\mu_N) = N^{-1} \log \mathbb{E}_J[(d\mathbb{P}_\beta/d\mathbb{P}_0) \circ \mu_N^{-1}]$
$\Rightarrow$ LDP for μ_N under $\mathbb{E}_J \otimes \mathbb{P}_\beta$, with rate $I(\mu) = 0 \Leftrightarrow \mu = \mu_\star$, yielding the LLN.

Limiting dynamics: Gaussian disorder, spherical model

Soft spherical spins: consider interacting Langevin diffusions

$$dx_t^{(i)} = -2U_\star'(\|\mathbf{x}_t\|^2/N)x_t^{(i)}dt + \frac{\beta}{\sqrt{N}}\sum_{j=1}^{N} J_{ij}x_t^{(j)}dt + dB_t^{(i)}, \quad (i = 1, \ldots, N),$$

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- Ben-Arous-D.-Guionnet (01'), show that uniformly on $[0, T]^2$:

$$C_N(s,t) = \frac{1}{N}\langle \mathbf{x}_s, \mathbf{x}_t\rangle \stackrel{a.s.}{\to} C_\infty(s,t)\,, \quad \text{as} \quad N \to \infty\,,$$

with C_∞ non-random, explicit, exhibiting FDT and AGING regimes (for $\beta > \beta_c$), as predicted by Cugliandolo-Dean (95').

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• Degenerate case of rich picture for limit dynamics of spherical mixed-spin models (see D.-Subag (20'), BenArous-Gheissari-Jagannath (20'), for analysis of generalized CK-CHS (93') Eqn.-s).

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• Similarly to [BG95],[G97], also in [BDG01],[DS20], etc., explicit Gaussian computations are the key.

Universality in spin glass models: static

Talagrand (06'), Gaussian disorder J, $\{\pm 1\}^N$;$\mathbb{S}^N$ valued, SK & mixed p-spins:

$$F_N(\mathsf{J}) := N^{-1} \log \int e^{\beta H_{\mathsf{J}}(\mathbf{x})} d\mathbf{x} \stackrel{a.s.}{\to} F_\beta \,, \quad \text{as} \quad N \to \infty \,,$$

with non-random F_β given by the corresponding Parisi formula.

Easy: Concentration, $|F_N(\mathsf{J}) - \mathbb{E} F_N| \to 0$ exponentially fast in N.

Hard: Convergence of $\mathbb{E} F_N$. Key: Gaussian integration by part.

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Chatterjee (05') [$\textsc{sk}$ on $\{\pm 1\}^N$]: $\mathsf{J} \mapsto F_N(\mathsf{J})$ smooth, small 3-rd derivatives,

$\Longrightarrow$ Lindeberg's principle applies (alt. see Carmona-Hu (06')).

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Heuristic: Entry-wise CLT for $\widehat{\mathbf{G}}_t = N^{-1/2}\widehat{\mathbf{J}}\mathbf{x}_t$ at reasonable, frozen $\mathbf{x}_t$.
$\Longrightarrow$ By Lindeberg's principle replace $\widehat{\mathbf{G}}_t$ with $\mathbf{G}_t = N^{-1/2}\mathbf{J}\mathbf{x}_t$ in (dsk_1)-$(dsk2)$.

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Challenge: $\mathbf{x}_t = \mathbf{x}_t(\mathrm{J})$ not frozen, potentially un-reasonable, no explicit solution.
$\Longrightarrow$ Not clear how to control (3rd order) derivatives of $\mathbf{G}_t(\mathrm{J})$, etc.

Girsanov & Lindeberg at the large deviations [DLZ19]

Compare two (dSK_1) diffusions of laws

$$\widehat{\mathbb{P}}_\beta : \qquad d\mathbf{x}_t = -\mathrm{diag}\{U'_\star(x_t^{(i)})\}dt + \frac{\beta}{\sqrt{N}}\widehat{\mathbf{J}}\mathbf{x}_t dt + d\mathbf{B}_t\,.$$

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$\Longrightarrow$ Just bound $\Delta_N = \widehat{\Gamma}_N - \Gamma_N$, for $\widehat{\Gamma}_N(\mu_N) = N^{-1}\log \mathbb{E}_{\widehat{\mathsf{J}}}[(d\widehat{\mathbb{P}}_\beta/d\mathbb{P}_0)\circ\mu_N^{-1}]$.

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- GIRSANOV and independence of rows of $\widehat{\mathsf{J}}$:

$\Longrightarrow$ $e^{N\widehat{\Gamma}_N} = \prod_i \mathbb{E}_{\widehat{\mathsf{J}}_i}[e^{-Q_i(\mathbf{x},\mathbf{B},\widehat{\mathsf{J}}_i)}]$, for $Q_i(\cdot) \geq 0$ explicit quadratic forms in $\widehat{\mathsf{J}}_i$.

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- RADON-NYKODIM: $(\mathbb{E}_J \otimes \widehat{\mathbb{P}}_\beta) \circ \mu_N^{-1}(A) = \mathbb{E}_\beta[e^{N\Delta_N}\mathbf{1}_{\{\mu_N \in A\}}]$

$\Longrightarrow$ Just bound $\Delta_N = \widehat{\Gamma}_N - \Gamma_N$, for $\widehat{\Gamma}_N(\mu_N) = N^{-1}\log \mathbb{E}_{\widehat{\mathsf{J}}}[(d\widehat{\mathbb{P}}_\beta / d\mathbb{P}_0) \circ \mu_N^{-1}]$.

- GIRSANOV and independence of rows of $\widehat{\mathsf{J}}$:

$\Longrightarrow$ $e^{N\widehat{\Gamma}_N} = \prod_i \mathbb{E}_{\widehat{\mathsf{J}}_i}[e^{-Q_i(\mathbf{x},\mathbf{B},\widehat{\mathsf{J}}_i)}]$, for $Q_i(\cdot) \geq 0$ explicit quadratic forms in $\widehat{\mathsf{J}}_i$.

- LINDEBERG: $\mathbb{E}_{\widehat{\mathsf{J}}_i}[e^{-Q_i}] \leq (1 + \frac{1}{\sqrt{N}}e^{M_i})\mathbb{E}_{\mathsf{J}_i}[e^{-Q_i}]$ with $M_i \approx \kappa \int_0^T (G_t^{(i)})^2 dt$.

$\Longrightarrow$ $\Delta_N \leq \frac{1}{N}\sum_{i=1}^N \log(1 + \frac{1}{\sqrt{N}}e^{M_i})$.

Girsanov & Lindeberg at the large deviations [DLZ19]

Compare two (dSK_1) diffusions of laws

$$\widehat{\mathbb{P}}_\beta : \qquad d\mathbf{x}_t = -\mathrm{diag}\{U'_\star(x_t^{(i)})\}dt + \frac{\beta}{\sqrt{N}}\widehat{\mathsf{J}}\mathbf{x}_t dt + d\mathbf{B}_t\,.$$

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Challenge (LD): Typically $M_i = O(1)$, but may be $O(N)$.
• Discretization & RMT: $\sum_i \widehat{\mathbb{P}}_\beta(\sum_i M_i \mathbf{1}_{\{M_i \geq r_N\}} \geq \eta N)$ finite, $\forall \eta > 0$, $r_N \to \infty$.

Stochastic Taylor expansion [DG20]

Denote by $\mathrm{P}_t^{(\widehat{\mathrm{J}})}$ the Markov semi-group of (dsk_2) at $U_\star(r) = \alpha r$:

$$d\mathbf{x}_t = -2\alpha \mathbf{x}_t dt + \widehat{\mathbf{G}}_t dt + d\mathbf{B}_t, \quad \widehat{\mathbf{G}}_t = \frac{\beta}{\sqrt{N}} \widehat{\mathrm{J}} \mathbf{x}_t, \quad \mathbf{x}_0 \sim \mu_0, \quad \mu_0 \text{ a product law}.$$

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Concentration for $f(\widehat{\mathrm{J}}, \mathbf{B}, \mathbf{x}_0)$, ex. $\mathbf{x}_0, \widehat{\mathrm{J}}$ satisfy POINCÁRE, $f(\cdot)$ is LIP.

$$\Longrightarrow \quad \text{Just show} \quad |\mathbb{E}_{\mathrm{J}}[\langle \mathrm{P}_t^{(\mathrm{J})} f, \mu_0 \rangle] - \mathbb{E}_{\widehat{\mathrm{J}}}[\langle \mathrm{P}_t^{(\widehat{\mathrm{J}})} f, \mu_0 \rangle]| \to 0 \text{ for } N \to \infty.$$

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After stochastic Taylor expansion

$$\langle \mathrm{P}_t^{(\widehat{\mathrm{J}})} f, \mu_0 \rangle = \sum_{k \geq 0} \frac{t^k}{k!} \langle (L^{(\widehat{\mathrm{J}})})^k f, \mu_0 \rangle,$$

suffices to show that as $N \to \infty$,

$$\sum_{k \geq 0} \frac{T^k}{k!} |\mathbb{E}_{\mathrm{J}}[\langle (L^{(\mathrm{J})})^k f, \mu_0 \rangle] - \mathbb{E}_{\widehat{\mathrm{J}}}[\langle (L^{(\widehat{\mathrm{J}})})^k f, \mu_0 \rangle]| \to 0. \qquad (\star)$$

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$L^{(\mathrm{J})} = \frac{\beta}{\sqrt{N}} \sum_{ij} J_{ji} x_i \partial_j - 2\alpha \sum_j x_j \partial_j + \sum_j \partial_{jj}$, so monomial $f(\mathrm{J}, \mathbf{x}) = \mathrm{J}_{\boldsymbol{\gamma}} \mathbf{x}_{\boldsymbol{\sigma}}$ yields $(L^{(\mathrm{J})})^k f$ a sum of monomials: $|\boldsymbol{\sigma}|$ non-increasing, contribution to $(\star)$ only if all multiplicities in $\boldsymbol{\gamma} \cup \widetilde{\boldsymbol{\gamma}}$ are ≥ 2, with one such ≥ 3.

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Thank you!